Second Order Linear Differential Equations

Second order linear equations with constant coefficients; Fundamental solutions; Wronskian; Existence and Uniqueness of solutions; the characteristic equation; solutions of homogeneous linear equations; reduction of order; Euler equations

In this chapter we will study ordinary differential equations of the standard form below, known as the *second order linear equations*:

$$\boxed{y'' + p(t)\,y' + q(t)\,y = g(t).}$$

Homogeneous Equations: If $g(t) = 0$, then the equation above becomes

$$y'' + p(t)\,y' + q(t)\,y = 0.$$

It is called a *homogeneous* equation. Otherwise, the equation is *nonhomogeneous* (or *inhomogeneous*).

Trivial Solution: For the homogeneous equation above, note that the function $y(t) = 0$ always satisfies the given equation, regardless what $p(t)$ and $q(t)$ are. This constant zero solution is called the *trivial solution* of such an equation.

Second Order Linear Homogeneous Differential Equations with Constant Coefficients

For the most part, we will only learn how to solve second order linear equation with constant coefficients (that is, when $p(t)$ and $q(t)$ are constants). Since a homogeneous equation is easier to solve compares to its nonhomogeneous counterpart, we start with second order linear homogeneous equations that contain constant coefficients only:

$$\boxed{ay'' + by' + cy = 0.}$$

Where a, b, and c are constants, $a \neq 0$.

A very simple instance of such type of equations is

$$y'' - y = 0.$$

The equation's solution is any function satisfying the equality $y'' = y$. Obviously $y_1 = e^t$ is a solution, and so is any constant multiple of it, $C_1 e^t$. Not as obvious, but still easy to see, is that $y_2 = e^{-t}$ is another solution (and so is any function of the form $C_2 e^{-t}$).

It can be easily verified that any function of the form

$$y = C_1 e^t + C_2 e^{-t}$$

will satisfy the equation. In fact, this is the general solution of the above differential equation.

Comment: Unlike first order equations we have seen previously, the general solution of a second order equation has *two* arbitrary coefficients.

Principle of Superposition: If y_1 and y_2 are any two solutions of the homogeneous equation

$$y'' + p(t)\,y' + q(t)\,y = 0.$$

Then any function of the form $y = C_1\,y_1 + C_2\,y_2$ is also a solution of the equation, for any pair of constants C_1 and C_2.

That is, for a homogeneous linear equation, any multiple of a solution is again a solution; any sum/difference of two solutions is again a solution; and the sum / difference of the multiples of any two solutions is again a solution. (This principle holds true for a homogeneous linear equation of *any* order; it is not a property limited only to a second order equation. It, however, **does not** hold, in general, for solutions of a nonhomogeneous linear equation.)

Note: However, while the general solution of $y'' + p(t)\,y' + q(t)\,y = 0$ will always be in the form of $C_1\,y_1 + C_2\,y_2$, where y_1 and y_2 are some solutions of the equation, the converse is not always true. Not every pair of solutions y_1 and y_2 could be used to give a general solution in the form $y = C_1\,y_1 + C_2\,y_2$. We shall see shortly the exact condition that y_1 and y_2 must satisfy that would give us a general solution of this form.

Fact: The general solution of a second order equation contains two arbitrary constants / coefficients. To find a particular solution, therefore, requires two initial values. The initial conditions for a second order equation will appear in the form: $y(t_0) = y_0$, and $y'(t_0) = y'_0$.

Question: Just by inspection, can you think of two (or more) functions that satisfy the equation $y'' + 4y = 0$? (Hint: A solution of this equation is a function φ such that $\varphi'' = -4\,\varphi$.)

Example: Find the general solution of

$$y'' - 5y' = 0.$$

There is no need to "guess" an answer here. We actually know a way to solve the equation already. Observe that if we let $u = y'$, then $u' = y''$. Substitute them into the equation and we get a new equation:

$$u' - 5\,u = 0.$$

This is a first order linear equation with $p(t) = -5$ and $g(t) = 0$. (!)

The integrating factor is $\mu = e^{-5t}$.

$$u(t) = \frac{1}{\mu(t)}\left(\int \mu(t) g(t)\, dt\right) = e^{5t}\left(\int 0\, dt\right) = e^{5t}(C) = Ce^{5t}$$

The actual solution y is given by the relation $u = y'$, and can be found by integration:

$$y(t) = \int u(t)\, dt = \int Ce^{5t}\, dt = \frac{C}{5}e^{5t} + C_2 = C_1\, e^{5t} + C_2\,.$$

The method used in the above example can be used to solve any second order linear equation of the form $y'' + p(t)\, y' = g(t)$, <u>regardless whether its coefficients are constant or nonconstant, or it is a homogeneous equation or nonhomogeneous.</u>

Equations of nonconstant coefficients with missing *y*-term

If the *y*-term (that is, the dependent variable term) is missing in a second order linear equation, then the equation can be readily converted into a first order linear equation and solved using the integrating factor method.

Example: $$t y'' + 4y' = t^2$$

The standard form is $y'' + \frac{4}{t} y' = t$.

Substitute: $u' + \frac{4}{t} u = t \qquad \rightarrow \qquad p(t) = \frac{4}{t}\ , g(t) = t$

Integrating factor is $\mu = t^4$.

$$u(t) = \frac{1}{t^4}\left(\int t^5\ dt\right) = t^{-4}\left(\frac{1}{6}t^6 + C\right) = \frac{1}{6}t^2 + C\,t^{-4}$$

Finally,

$$y(t) = \int u(t)\,dt = \frac{1}{18}t^3 - \frac{C}{3}t^{-3} + C_2 = \frac{1}{18}t^3 + C_1\,t^{-3} + C_2$$

Comment: Notice the above solution is not in the form of $y = C_1\,y_1 + C_2\,y_2$. There is nothing wrong with this, because this equation is not homogeneous. The general solution of a nonhomogeneous linear equation has a slightly different form. We will learn about the solutions of nonhomogeneous linear equations a bit later.

In general, given a second order linear equation with the y-term missing

$$y'' + p(t)\,y' = g(t),$$

we can solve it by the substitutions $u = y'$ and $u' = y''$ to change the equation to a first order linear equation. Use the integrating factor method to solve for u, and then integrate u to find y. That is:

1. Substitute:	$u' + p(t)\,u = g(t)$
2. Integrating factor:	$\mu(t) = e^{\int p(t)\,dt}$
3. Solve for u:	$u(t) = \dfrac{\int \mu(t)g(t)\,dt \;\left(+\,C\right)}{\mu(t)}$
4. Integrate:	$y(t) = \int u(t)\;dt$

This method works regardless whether the coefficients are constant or nonconstant, or if the equation is nonhomogeneous.

The Characteristic Polynomial

Back to the subject of the second order linear homogeneous equations with constant coefficients (note that it is not in the standard form below):

$$a y'' + b y' + c y = 0, \qquad a \neq 0. \qquad (*)$$

We have seen a few examples of such an equation. In all cases the solutions consist of exponential functions, or terms that could be rewritten into exponential functions†. With this fact in mind, let us derive a (very simple, as it turns out) method to solve equations of this type. We will start with the assumption that there are indeed some exponential functions of unknown exponents that would satisfy any equation of the above form. We will then devise a way to find the specific exponents that would give us the solution.

Let $y = e^{rt}$ be a solution of (*), for some as-yet-unknown constant r. Substitute y, $y' = re^{rt}$, and $y'' = r^2 e^{rt}$ into (*), we get

$$a r^2 e^{rt} + b r e^{rt} + c e^{rt} = 0, \qquad \text{or}$$

$$e^{rt}(a r^2 + b r + c) = 0.$$

Since e^{rt} is never zero, the above equation is satisfied (and therefore $y = e^{rt}$ is a solution of (*)) if and only if $a r^2 + b r + c = 0$. Notice that the expression $a r^2 + b r + c$ is a quadratic polynomial with r as the unknown. It is always solvable, with roots given by the quadratic formula. Hence, we can always solve a second order linear homogeneous equation with constant coefficients (*).

† Sine and cosine are related to exponential functions by the identities

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

This polynomial, $ar^2 + br + c$, is called the *characteristic polynomial* of the differential equation (*). The equation

$$\boxed{ar^2 + br + c = 0}$$

is called the *characteristic equation* of (*). Each and every root, sometimes called a *characteristic root*, r, of the characteristic polynomial gives rise to a solution $y = e^{rt}$ of (*).

We will take a more detailed look of the 3 possible cases of the solutions thusly found:

1. (When $b^2 - 4ac > 0$) There are two distinct real roots r_1, r_2.
2. (When $b^2 - 4ac < 0$) There are two complex conjugate roots $r = \lambda \pm \mu i$.
3. (When $b^2 - 4ac = 0$) There is one repeated real root r.

Note: There is no need to put the equation in its standard form when solving it using the characteristic equation method. The roots of the characteristic equation remain the same regardless whether the leading coefficient is 1 or not.

Case 1 Two distinct real roots

When $b^2 - 4ac > 0$, the characteristic polynomial have two distinct real roots r_1, r_2. They give two distinct[‡] solutions $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$. Therefore, a general solution of (*) is

$$\boxed{y = C_1 y_1 + C_2 y_2 = C_1 e^{r_1 t} + C_2 e^{r_2 t}.}$$

It is **that** easy.

Example: $y'' + 5y' + 4y = 0$

The characteristic equation is $r^2 + 5r + 4 = (r + 1)(r + 4) = 0$, the roots of the polynomial are $r = -1$ and -4. The general solution is then

$$y = C_1 e^{-t} + C_2 e^{-4t}.$$

Suppose there are initial conditions $y(0) = 1$, $y'(0) = -7$. A unique particular solution can be found by solving for C_1 and C_2 using the initial conditions. First we need to calculate $y' = -C_1 e^{-t} - 4C_2 e^{-4t}$, then apply the initial values:

$$1 = y(0) = C_1 e^0 + C_2 e^0 = C_1 + C_2$$

$$-7 = y'(0) = -C_1 e^0 - 4C_2 e^0 = -C_1 - 4C_2$$

The solution is $C_1 = -1$, and $C_2 = 2$ $\rightarrow$ $y = -e^{-t} + 2e^{-4t}$.

‡ We shall see the precise meaning of distinctness in the next section. For now just think that the two solutions are not constant multiples of each other.

Question: Suppose the initial conditions are instead $y(10000) = 1$, $y'(10000) = -7$. How would the new t_0 change the particular solution?

Apply the initial conditions as before, and we see there is a little complication. Namely, the simultaneous system of 2 equations that we have to solve in order to find C_1 and C_2 now comes with rather inconvenient irrational coefficients:

$$1 = y(10000) = C_1 e^{-10000} + C_2 e^{-40000}$$

$$-7 = y'(10000) = -C_1 e^{-10000} - 4C_2 e^{-40000}$$

With some good bookkeeping, systems like this can be solved the usual way. However, there is an easier method to simplify the inconvenient coefficients. The idea is *translation* (or *time-shift*). What we will do is to first construct a new coordinate axis, say $\check{T}$-axis. The two coordinate-axes are related by the equation $\check{T} = t - t_0$. (Therefore, when $t = t_0$, $\check{T} = 0$; that is, the initial t-value t_0 becomes the new origin.) In other words, we translate (or time-shift) t-axis by t_0 units to make it $\check{T}$-axis. In this example, we will accordingly set $\check{T} = t - 10000$. The immediate effect is that it makes the initial conditions to be back at 0: $y(0) = 1$, $y'(0) = -7$, with respect to the new $\check{T}$-coordinate. We then solve the translated system of 2 equations to find C_1 and C_2. What we get is the (simpler) system

$$1 = y(0) = C_1 e^{0} + C_2 e^{0} = C_1 + C_2$$

$$-7 = y'(0) = -C_1 e^{0} - 4C_2 e^{0} = -C_1 - 4C_2$$

As we have seen on the previous page, the solution is $C_1 = -1$, and $C_2 = 2$. Hence, the solution, in the new $\check{T}$-coordinate system, is $y(\check{T}) = -e^{-\check{T}} + 2e^{-4\check{T}}$.

Lastly, since this solution is in terms of $\check{T}$, but the original problem was in terms of t, we should convert it back to the original context. This conversion is easily achieved using the translation formula used earlier, $\check{T} = t - t_0 = t - 10000$. By replacing every occurrence of $\check{T}$ by $t - 1000$ in the solution, we obtain the solution, in its proper independent variable t.

$$y(t) = -e^{-(t - 10000)} + 2e^{-4(t - 10000)}.$$

Example: Consider the solution $y(t)$ of the initial value problem

$$y'' - 2y' - 8y = 0, \qquad\qquad y(0) = \alpha, \quad y'(0) = 2\pi.$$

Depending on the value of α, as $t \to \infty$, there are 3 possible behaviors of $y(t)$. Explicitly determine the possible behaviors and the respective initial value α associated with each behavior.

The characteristic equation is $r^2 - 2r - 8 = (r + 2)(r - 4) = 0$. Its roots are $r = -2$ and 4. The general solution is then

$$y = C_1 e^{-2t} + C_2 e^{4t}.$$

Notice that the long-term behavior of the solution is dependent on the coefficient C_2 only, since the $C_1 e^{-2t}$ term tends to 0 as $t \to \infty$, regardless of the value of C_1.

Solving for C_2 in terms of α, we get

$$\begin{aligned} y(0) &= \alpha &&= C_1 + C_2 \\ y'(0) &= 2\pi &&= -2C_1 + 4C_2 \end{aligned}$$

$$2\alpha + 2\pi = 6C_2 \qquad \to \qquad C_2 = \frac{\alpha + \pi}{3}.$$

Now, if $C_2 > 0$ then y tends to ∞ as $t \to \infty$. This would happen when $\alpha > -\pi$. If $C_2 = 0$ then y tends to 0 as $t \to \infty$. This would happen when $\alpha = -\pi$. Lastly, if $C_2 < 0$ then y tends to $-\infty$ as $t \to \infty$. This would happen when $\alpha < -\pi$. In summary:

When $\alpha > -\pi$,	$C_2 > 0$,	$\lim_{t\to\infty} y(t) = \infty$.
When $\alpha = -\pi$,	$C_2 = 0$,	$\lim_{t\to\infty} y(t) = 0$.
When $\alpha < -\pi$,	$C_2 < 0$,	$\lim_{t\to\infty} y(t) = -\infty$.

The Existence and Uniqueness (of the solution of a second order linear equation initial value problem)

A sibling theorem of the first order linear equation Existence and Uniqueness Theorem…

Theorem: Consider the initial value problem

$$y'' + p(t)\, y' + q(t)\, y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y'_0.$$

If the functions p, q, and g are continuous on the interval $\boldsymbol{I}$: $\alpha < t < \beta$ containing the point $t = t_0$. Then there exists a unique solution $y = \varphi(t)$ of the problem, and that this solution exists throughout the interval $\boldsymbol{I}$.

That is, the theorem guarantees that the given initial value problem will always have (existence of) exactly one (uniqueness) twice-differentiable solution, on any interval containing t_0 as long as all three functions $p(t)$, $q(t)$, and $g(t)$ are continuous on the same interval. Conversely, neither existence nor uniqueness of a solution is guaranteed at a discontinuity of $p(t)$, $q(t)$, or $g(t)$.

Examples: For each IVP below, find the largest interval on which a unique solution is guaranteed to exist.

(a) $(t+2)y'' + t y' + \cot(t) y = t^2 + 1, \qquad y(2) = 11, \; y'(2) = -2.$

The standard form is $y'' + \dfrac{t}{t+2} y' + \dfrac{\cos(t)}{(t+2)\sin(t)} y = \dfrac{t^2+1}{t+2}$, and $t_0 = 2$. The discontinuities of p, q, and g are $t = -2, 0, \pm\pi, \pm 2\pi, \pm 3\pi \ldots$ The largest interval that contains $t_0 = 2$ but none of the discontinuities is, therefore, $(0, \pi)$.

(b) $\sqrt{16-t^2}\, y'' + \ln(t+1) y' + \cos(t) y = 0$, $y(0) = 2$, $y'(0) = 0$.

The standard form is $y'' + \dfrac{\ln(t+1)}{\sqrt{16-t^2}} y' + \dfrac{\cos(t)}{\sqrt{16-t^2}} y = 0$, $p(t)$ is only defined (and is continuous) on the interval $(-1, 4)$, and similarly $q(t)$ is only continuously defined on the interval $(-4, 4)$; $g(t)$ is continuous everywhere. Combining them we see that p, q, and g have discontinuities at any t such that $t \le -1$ or $t \ge 4$. That is, they are all continuous only on the interval $(-1, 4)$. Since that interval contains $t_0 = 0$, it must be the largest interval on which the solution is guaranteed to exist uniquely. Therefore, the answer is $(-1, 4)$

Similar to the previous instance (first order linear equation version) of the Existence and Uniqueness Theorem, the only time that a unique solution is not guaranteed to exist anywhere is whenever the initial time t_0 occurs at a discontinuity of either $p(t)$, $q(t)$, or $g(t)$.

Initial Value Problem vs. Boundary Value Problem

It might seem that there are more than one ways to present the initial conditions of a second order equation. Instead of locating both initial conditions $y(t_0) = y_0$ and $y'(t_0) = y'_0$ at the same point t_0, couldn't we take them at different points, for examples $y(t_0) = y_0$ and $y(t_1) = y_1$; or $y'(t_0) = y'_0$ and $y'(t_1) = y'_1$? The answer is NO. All the initial conditions in an initial value problem <u>must</u> be taken at the same point t_0. The sets of conditions above where the values are taken at different points are known as *boundary conditions*. A boundary value problem where a differential equation is bundled with (two or more) boundary conditions does not have the existence and uniqueness guarantee.

Example: Every function of the form $y = C\sin(t)$, where C is a real number satisfies the boundary value problem $y'' + y = 0$, $y(0) = 0$ and $y(\pi) = 0$. Therefore, the problem has infinitely many solutions, even though $p(t) = 0$, $q(t) = 1$, and $g(t) = 0$ are all continuous everywhere.

Exercises B.1-1:

1 – 4 Find the general solution of each equation.

1. $y'' + 10y' = t^2$

2. $y'' - 9y = 0$

3. $y'' + 4y' - 5y = 0$

4. $6y'' + y' - y = 0$

5 – 9 Solve each initial value problem. For each problem, state the largest interval in which the solution is guaranteed to uniquely exist.

5. $y'' + y' = 3e^{t/2}$, $\quad y(0) = 4$, $\quad y'(0) = 3$

6. $y'' + 2y' = t\,e^{-t}$, $\quad y(0) = 6$, $\quad y'(0) = -1$

7. $ty'' - y' = t^2 + t$, $\quad y(1) = 1$, $\quad y'(1) = 5$

8. $y'' - y' - 2y = 0$, $\quad y(0) = 2$, $\quad y'(0) = 7$

9. $(t^2 + 9)\,y'' + 2ty' = 0$, $\quad y(3) = 2\pi$, $\quad y'(3) = 2/3$

10 – 15 Solve each initial value problem.

10. $y'' + y' - 12y = 0$, $\quad y(0) = -2$, $\quad y'(0) = -20$

11. $y'' + y' - 12y = 0$, $\quad y(\pi) = -2$, $\quad y'(\pi) = -20$

12. $y'' + 2y' - 3y = 0$, $\quad y(0) = 1$, $\quad y'(0) = 13$

13. $y'' + 2y' - 3y = 0$, $\quad y(2\pi) = 1$, $\quad y'(2\pi) = 13$

14. $y'' + 2y' - 4y = 0$, $\quad y(0) = 6$, $\quad y'(0) = -6$

15. $y'' + 2y' - 4y = 0$, $\quad y(18) = 6$, $\quad y'(18) = -6$

16. Without solving the given initial value problem, what is the largest interval in which a unique solution is guaranteed to exist?

$$(t + 10)\,y'' - (5 - t)\,y' + \ln|\,t\,|\,y = e^{2t}\cos t,$$

(a) $y(1) = -1, \quad y'(1) = 0$
(b) $y(-9) = 3, \quad y'(-9) = -2$
(c) $y(-12.5) = 1, \quad y'(-12.5) = 4$

17. Prove the Principle of Superposition: If y_1 and y_2 are any two solutions of the homogeneous equation

$$y'' + p(t)\,y' + q(t)\,y = 0.$$

Then any function of the form $y = C_1\,y_1 + C_2\,y_2$ is also a solution of the equation, for any pair of constants C_1 and C_2.

Answers B-1.1:

1. $y = \dfrac{t^3}{30} - \dfrac{t^2}{100} + \dfrac{t}{500} + C_1 e^{-10t} + C_2$
2. $y = C_1 e^{3t} + C_2 e^{-3t}$
3. $y = C_1 e^{t} + C_2 e^{-5t}$
4. $y = C_1 e^{t/3} + C_2 e^{-t/2}$
5. $y = -e^{-t} + 1 + 4e^{t/2}, \quad (-\infty, \infty)$
6. $y = -t e^{-t} + 6, \quad (-\infty, \infty)$
7. $y = \dfrac{t^3}{3} + \dfrac{7t^2}{4} + \dfrac{t^2}{2}\ln t - \dfrac{13}{12}, \qquad (0, \infty)$
8. $y = 3e^{2t} - e^{-t}, \quad (-\infty, \infty)$
9. $y = 4\tan^{-1}\left(\dfrac{t}{3}\right) + \pi, \qquad (-\infty, \infty)$
10. $y = -4e^{3t} + 2e^{-4t}$
11. $y = -4e^{3(t-\pi)} + 2e^{-4(t-\pi)}$
12. $y = 4e^{t} - 3e^{-3t}$
13. $y = 4e^{t-2\pi} - 3e^{-3(t-2\pi)}$
14. $y = 3e^{(-1+\sqrt{5})t} + 3e^{(-1-\sqrt{5})t}$
15. $y = 3e^{(-1+\sqrt{5})(t-18)} + 3e^{(-1-\sqrt{5})(t-18)}$
16. (a) $(0, \infty)$, (b) $(-10, 0)$, (c) $(-\infty, -10)$

Fundamental Solutions

We have seen that the general solution of a second order homogeneous linear equation is in the form of $y = C_1 y_1 + C_2 y_2$ [§], where y_1 and y_2 are two "distinct" functions both satisfying the given equation (as a result, y_1 and y_2 are themselves particular solutions of the equation). Now we will examine the circumstance under which two arbitrary solutions y_1 and y_2 could give us a general solution.

Suppose y_1 and y_2 are two solutions of some second order homogeneous linear equation such that their linear combinations $y = C_1 y_1 + C_2 y_2$ give a general solution of the equation. Then, according to the Existence and Uniqueness Theorem, for any pair of initial conditions $y(t_0) = y_0$ and $y'(t_0) = y'_0$ there must exist uniquely a corresponding pair of coefficients C_1 and C_2 that satisfies the system of (algebraic) equations

$$\begin{aligned} y_0 &= C_1\, y_1(t_0) + C_2\, y_2(t_0) \\ y'_0 &= C_1\, y'_1(t_0) + C_2\, y'_2(t_0) \end{aligned}$$

From linear algebra, we know that for the above system to always have a unique solution (C_1, C_2) for any initial values y_0 and y'_0, the coefficient matrix of the system must be invertible, or, equivalently, <u>the determinant of the coefficient matrix must be nonzero[**]</u>. That is

$$\det\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y'_1(t_0) & y'_2(t_0) \end{pmatrix} = y_1(t_0)\,y'_2(t_0) - y'_1(t_0)\,y_2(t_0) \neq 0$$

This determinant above is called the *Wronskian* or the *Wronskian determinant*. It is a function of t as well, denoted $W(y_1, y_2)(t)$, and is given by the expression

$$\boxed{W(y_1, y_2)(t) = y_1\, y'_2 - y'_1\, y_2.}$$

[§] The expression $y = C_1 y_1 + C_2 y_2$ is called a *linear combination* of the functions y_1 and y_2.

[**] By nonzero it means that the Wronskian is not the constant zero function.

On the other hand, at each point t_0 where $W(y_1, y_2)(t_0) = 0$, a unique pair of coefficients C_1 and C_2 that satisfies the previous system of equations cannot always be found (see any linear algebra textbook for a proof of this). This could be due to one of two reasons. The first reason is that $y = C_1\, y_1 + C_2\, y_2$ is really not a general solution of our equation. Or, the second possibility is that t_0 is a discontinuity of either $p(t)$, $q(t)$, or $g(t)$. This second reason is, of course, a consequence of the Existence and Uniqueness theorem.

Assuming that not every point is a discontinuity of either $p(t)$, $q(t)$, or $g(t)$, then the fact that $W(y_1, y_2)(t)$ is constant zero implies that $y = C_1\, y_1 + C_2\, y_2$ is not a general solution of the given equation. Otherwise, if $W(y_1, y_2)(t)$ is nonzero at some points t_0 on the real line, then $y = C_1\, y_1 + C_2\, y_2$ will, together with different combinations of initial condition $y(t_0) = y_0$ and $y'(t_0) = y'_0$, give uniquely all the possible particular solutions, on some open intervals containing t_0. That is, $y = C_1\, y_1 + C_2\, y_2$ is a general solution of the given equation. Hence, our interest in knowing whether or not $W(y_1, y_2)(t)$ is the constant zero function.

Formally, if $W(y_1, y_2)(t) \neq 0$, then the functions y_1, y_2 are said to be *linearly independent*. Else they are called *linearly dependent* if $W(y_1, y_2)(t) = 0$.[††]

Note: In the simple instance of two functions, as is the case presently, their linear independence could equivalently be determined by the fact that two functions are linearly independent if and only if they are not constant multiples of each other.

Suppose y_1 and y_2 are two linearly independent solutions of a second order homogeneous linear equation

$$y'' + p(t)\, y' + q(t)\, y = 0.$$

That is, y_1 and y_2 both satisfy the equation, and $W(y_1, y_2)(t) \neq 0$. Then (and only then) their linear combination $y = C_1\, y_1 + C_2\, y_2$ forms a general solution of the differential equation. Therefore, a pair of such linearly independent solutions y_1 and y_2 is called a set of *fundamental solutions*, because they are

[††] Since $W(y_1, y_2)(t) = -W(y_2, y_1)(t)$, they are either both zero or both nonzero. Therefore, the order of the 2 functions y_1 and y_2 does not matter in the Wronskian calculation.

essentially the basic building blocks of all particular solutions of the equation.

To summarize, suppose y_1 and y_2 are two solutions of a second order homogeneous linear equation, then:

$W(y_1, y_2)(t)$ is not the constant zero function

$\updownarrow$

y_1, y_2 are linearly independent

$\updownarrow$

y_1, y_2 are fundamental solutions

$\updownarrow$

$y = C_1 y_1 + C_2 y_2$ is a general solution of the equation

Example: Let $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$, $r_1 \neq r_2$, be any two different exponential function. Then

$$W(y_1, y_2) = \det\begin{pmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{pmatrix} = r_2 e^{r_1 t} e^{r_2 t} - r_1 e^{r_1 t} e^{r_2 t}$$

$$= (r_2 - r_1) e^{r_1 t} e^{r_2 t} \neq 0, \qquad \text{for all } t.$$

Therefore, any two different exponential-function solutions of a second order homogeneous linear equation (as those found using its characteristic equation) are always linearly independent, thus they will always give a general solution. Better yet, in this case since the Wronskian is never zero for all real numbers, a unique solution can always be found.

Lastly, here is an interesting (and, as we shall see shortly, useful) relationship between the Wronskian of any two solutions of a second order linear equation with its coefficient function $p(t)$.

Abel's Theorem: If y_1 and y_2 are any two solutions of the equation

$$y'' + p(t)\,y' + q(t)\,y = 0,$$

where p and q are continuous on an open interval $\boldsymbol{I}$. Then the Wronskian $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = C\,e^{-\int p(t)\,dt},$$

where C is a constant that depends on y_1 and y_2, but not on t. Further, $W(y_1, y_2)(t)$ is either zero for all t in $\boldsymbol{I}$ (if $C = 0$) or else is never zero in $\boldsymbol{I}$ (if $C \neq 0$).

Exercises B-1.2:

1. Previously, we have found that the equation $y'' - y = 0$ has a general solution $y = C_1 e^{t} + C_2 e^{-t}$. (a) Construct another general solution by first verifying that $y_1 = \cosh t = \dfrac{e^t + e^{-t}}{2}$ and $y_2 = \sinh t = \dfrac{e^t - e^{-t}}{2}$ also form a pair of fundamental solutions. Conclude that a general solution is not unique for this equation. (b) For each of the two general solutions, find the solution corresponding to the initial conditions $y(0) = 1$ and $y'(0) = 2$. Show that the two particular solutions are identical.

2. Suppose y_1 and y_2 are two solutions of the equation $t^2 y'' + 2t^3 y' - t^{-2} y = 0$. Find $\boldsymbol{W}(y_1, y_2)(t)$.

3. Suppose y_1 and y_2 are two solutions of the equation $t\,y'' - (t + 4)\,y' + e^{-3t} y = 0$, such that $\boldsymbol{W}(y_1, y_2)(1) = 10$. Find $\boldsymbol{W}(y_1, y_2)(t)$.

4. Suppose $y_1 = t$ and $y_2 = t\,e^{4t}$ are both solutions of a certain equation $y'' + p(t)\,y' + q(t)\,y = 0$. (a) Compute $\boldsymbol{W}(y_1, y_2)(t)$. (b) What is a general solution of this equation? (c) Does there exist a unique solution satisfying the initial conditions $y(0) = 0, y'(0) = 0$? (Use part b in your computation, is there a unique pair of coefficients C_1 and C_2?) (d) Find the solution satisfying the initial conditions $y(1) = 1, y'(1) = 5$. (e) What is the largest interval on which the solution from part d is guaranteed to exist uniquely?

5. Suppose $y_1 = 2 + 3\,e^{-t}$ and $y_2 = 3 - 2\,e^{-t}$ are both solutions of a certain equation $y'' + p(t)\,y' + q(t)\,y = 0$. (a) Compute $\boldsymbol{W}(y_1, y_2)(t)$. (b) What is a general solution of this equation? (c) Find the solution satisfying the initial conditions $y(0) = 2, y'(0) = 3$. (d) What is the largest interval on which the solution from part d is guaranteed to exist uniquely?

Answers B-1.2:

1. (a) Another general solution is $y = C_1 \cosh t + C_2 \sinh t$.
2. $\boldsymbol{W}(y_1, y_2)(t) == Ce^{-t^2}$
3. $\boldsymbol{W}(y_1, y_2)(t) = 10t^4 e^{t-1}$
4. (a) $\boldsymbol{W}(y_1, y_2)(t) = 4t^2 e^{4t}$, (b) $y = C_1 t + C_2 t e^{4t}$, (c) Since $\boldsymbol{W}(y_1, y_2)(0) = 0$, there is no existence or uniqueness guarantee for a particular solution. As it turns out, there are infinitely many solutions satisfying the given initial conditions: any function of the form $y = C_1 t + C_2 t e^{4t}$, where $C_1 = -C_2$. (d) $y = e^{-4} t e^{4t}$, (e) $(0, \infty)$.
5. (a) $\boldsymbol{W}(y_1, y_2)(t) = 13e^{-t}$, (b) $y = C_1 (2 + 3e^{-t}) + C_2 (3 - 2e^{-t})$, which can be simplified to $y = K_1 + K_2 e^{-t}$, (c) $y = 5 - 3e^{-t}$, (d) $(-\infty, \infty)$.

<u>Case 2</u> Two complex conjugate roots

When $b^2 - 4ac < 0$, the characteristic polynomial has two complex roots, which are conjugates, $r_1 = \lambda + \mu i$ and $r_2 = \lambda - \mu i$ (λ, μ are real numbers, $\mu > 0$). As before they give two linearly independent solutions $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$. Consequently the linear combination $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ will be a general solution. At this juncture you might have this question: "but aren't r_1 and r_2 complex numbers; what would become of the exponential function with a complex number exponent?" The answer to that question is given by the *Euler's formula*.

> ***Euler's formula*** For any real number θ,
>
> $$e^{\theta i} = \cos\theta + i\sin\theta\,.$$

Hence, when r is a complex number $\lambda + \mu i$, the exponential function e^{rt} becomes

$$e^{rt} = e^{(\lambda + \mu i)t} = e^{\lambda t} e^{\mu i t} = e^{\lambda t}(\cos \mu t + i \sin \mu t)$$

Similarly, when $r = \lambda - \mu i$, e^{rt} becomes

$$\begin{aligned} e^{(\lambda - \mu i)t} &= e^{\lambda t} e^{-\mu i t} = e^{\lambda t}(\cos(-\mu t) + i \sin(-\mu t)) \\ &= e^{\lambda t}(\cos \mu t - i \sin \mu t) \end{aligned}$$

Hence, the general solution found above is then

$$y = C_1\, e^{\lambda t}(\cos \mu t + i \sin \mu t) + C_2\, e^{\lambda t}(\cos \mu t - i \sin \mu t)$$

However, this general solution is a complex-valued function (meaning that, given a real number t, the value of the function $y(t)$ could be complex). It represents the general form of all particular solutions with either real or complex number coefficients. What we seek here, instead, is a real-valued expression that gives only the set of all particular solutions with real number coefficients only. In other words, we would like to "filter out" all functions containing coefficients with an imaginary part, that satisfy the given differential equation, keeping only those whose coefficients are real numbers.

Define
$$u(t) = e^{\lambda t} \cos \mu t$$
$$v(t) = e^{\lambda t} \sin \mu t$$

It is easy to verify that both u and v satisfy the differential equation (one way to see this is to observe that u can be obtain from the complex-valued general solution by setting $C_1 = C_2 = 1/2$; and v can be obtained similarly by setting $C_1 = 1/2i$ and $C_2 = -1/2i$). Their Wronskian is $W(u, v) = \mu e^{2\lambda t}$ is never zero. Therefore, the functions u and v are linearly independent solutions of the equation. They form a pair of real-valued fundamental solutions and the linear combination is a desired real-valued general solution:

$$\boxed{y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t.}$$

When $r = \lambda \pm \mu i$, $\mu > 0$, are two complex roots of the characteristic polynomial.

Example: $y'' + 4y = 0$

Answer: $y = C_1 \cos 2t + C_2 \sin 2t$

Example: $y'' + 2y' + 5y = 0, \qquad y(0) = 4, \quad y'(0) = 6$

The characteristic equation is $r^2 + 2r + 5 = 0$, which has solutions $r = -1 \pm 2i$. So $\lambda = -1$ and $\mu = 2$. Therefore, the general solution is

$$y = C_1 e^{-t} \cos 2t + C_2 e^{-t} \sin 2t$$

Apply the initial conditions to find that $C_1 = 4$ and $C_2 = 5$. Hence,

$$y = 4e^{-t} \cos 2t + 5e^{-t} \sin 2t.$$

Question: What would the solution be if the initial conditions are $y(25000) = 4$, and $y'(25000) = 6$ instead?

Answer: $y = 4e^{-(t-25000)} \cos 2(t - 25000) + 5e^{-(t-25000)} \sin 2(t - 25000)$

Case 3 One repeated real root

When $b^2 - 4ac = 0$, the characteristic polynomial has a single repeated real root, $r = \dfrac{-b}{2a}$. This causes a problem, because unlike the previous two cases the roots of characteristic polynomial presently only give us one distinct solution $y_1 = e^{rt}$. It is not enough to give us a general solution. We would need to come up with a second solution, linearly independent with y_1, on our own. How do we find a second solution?

Take what we have: a solution $y_1 = e^{rt}$, where $r = \dfrac{-b}{2a}$. Let y_2 be another solution of the same equation $ay'' + by' + cy = 0$. The standard form of this equation is $y'' + \dfrac{b}{a}y' + \dfrac{c}{a}y = 0$, where $p(t) = \dfrac{b}{a}$. Compute the Wronskian two different ways:

$$W(y_1, y_2) = \det\begin{pmatrix} e^{rt} & y_2 \\ re^{rt} & y_2' \end{pmatrix} = e^{rt}y_2' - re^{rt}y_2, \quad r = \frac{-b}{2a}$$

and

$$W(y_1, y_2) = C\,e^{-\int p(t)\,dt} = C\,e^{-\int \frac{b}{a}dt} = C\,e^{\frac{-b}{a}t}, \qquad C \neq 0.$$

By the Abel's Theorem, the fact $C \neq 0$ guarantees that y_1 and y_2 are going to be linearly independent. Now, we have two expressions for the Wronskian of the same pair of solutions. The two expressions must be equal:

$$e^{\frac{-b}{2a}t}\,y_2' + \frac{b}{2a}e^{\frac{-b}{2a}t}\,y_2 = C\,e^{\frac{-b}{a}t}, \qquad C \neq 0.$$

This is a first order linear differential equation with y_2 as the unknown!

Put it into its standard form and solve by the integrating factor method.

$$y_2' + \frac{b}{2a} y_2 = C e^{\frac{-b}{2a}t}$$

The integrating factor is $\mu = e^{\int \frac{b}{2a} dt} = e^{\frac{b}{2a}t}$.

Hence,

$$y_2 = \frac{1}{e^{\frac{b}{2a}t}} \int e^{\frac{b}{2a}t} C e^{\frac{-b}{2a}t} dt = e^{\frac{-b}{2a}t} \int C\, dt = e^{\frac{-b}{2a}t} (Ct + C_1)$$

$$= Cte^{\frac{-b}{2a}t} + C_1 e^{\frac{-b}{2a}t} = Cte^{rt} + C_1 e^{rt}$$

Any such a function would be a second, linearly independent solution of the differential equation. We just need one instance of such a function. The only condition for the coefficients in the above expression is $C \neq 0$. Pick, say, $C = 1$, and $C_1 = 0$ would work nicely. Thus $y_2 = t\, e^{rt}$.

Therefore, the general solution in the case of a repeated real root r is

$$\boxed{y = C_1 e^{rt} + C_2 t e^{rt}.}$$

Example: $y'' - 4y' + 4y = 0, \qquad y(0) = 4, \quad y'(0) = 5$

The characteristic equation is $r^2 - 4r + 4 = (r - 2)^2 = 0$, which has solution $r = 2$ (repeated). Thus, the general solution is

$$y = C_1 e^{2t} + C_2 t e^{2t}.$$

Differentiate,

$$y' = 2C_1 e^{2t} + C_2 (2t e^{2t} + e^{2t}).$$

Apply the initial conditions to find that $C_1 = 4$ and $C_2 = -3$:

$$y = 4 e^{2t} - 3t e^{2t}.$$

Summary

Given a second order linear equation with constant coefficients

$$a y'' + b y' + c y = 0, \qquad a \neq 0.$$

Solve its characteristic equation $a r^2 + b r + c = 0$. The general solution depends on the type of roots obtained (use the quadratic formula to find the roots if you are unable to factor the polynomial!):

1. When $b^2 - 4ac > 0$, there are two distinct real roots r_1, r_2.

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

2. When $b^2 - 4ac < 0$, there are two complex conjugate roots $r = \lambda \pm \mu i$. Then

$$y = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t.$$

3. When $b^2 - 4ac = 0$, there is one repeated real root r. Then

$$y = C_1 e^{rt} + C_2 t e^{rt}.$$

Since $p(t) = b/a$ and $q(t) = c/a$, being constants, are continuous for every real number, therefore, according to the Existence and Uniqueness Theorem, in each case above there is always a unique solution valid on $(-\infty, \infty)$ for any pair of initial conditions $y(t_0) = y_0$ and $y'(t_0) = y'_0$.

Exercises B-1.3:

1. Verify that $y = t\,e^{rt}$, $r = \dfrac{-b}{2a}$, is a solution of the equation $a\,y'' + b\,y' + c\,y = 0$ if $b^2 - 4\,ac = 0$; and it is not a solution if $b^2 - 4\,ac \neq 0$.

2 − 10 For each of the following equations (a) find its general solution, (b) find the particular solution satisfying the initial conditions $y(0) = 2$, $y'(0) = -1$, and (c) find the limit, as $t \to \infty$, of the solution found in (b).

2. $y'' + 9y' + 8y = 0$

3. $y'' - 6y' + 25y = 0$

4. $y'' - 6y' + 8y = 0$

5. $2y'' + 5y' - 3y = 0$

6. $2y'' - 16y' + 32y = 0$

7. $y'' + 4y' + 13y = 0$

8. $2y'' - y' = 0$

9. $2y'' + 5y' + 2y = 0$

10. $16y'' - 8y' + y = 0$

11 – 15 Solve each initial value problem.

11. $y'' + 9y' + 14y = 0$, $\quad y(5\pi) = 4$, $\quad y'(5\pi) = 2$

12. $2y'' - 16y' + 32y = 0$, $\quad y(-2) = 2$, $\quad y'(-2) = -1$

13. $9y'' + y = 0$, $\quad y(0) = -2$, $\quad y'(0) = 2$

14. $y'' + 6y' + 34y = 0$, $\quad y(10) = 5$, $\quad y'(10) = -5$

15. $10y'' - 7y' + y = 0$, $\quad y(0) = -8$, $\quad y'(0) = -1$

16 – 21 Find a second order linear equation with constant coefficients that has the indicated solution. (The answer is not unique.)

16. The general solution is $y = C_1 e^{t} + C_2 t e^{t}$.

17. The general solution is $y = C_1 e^{5t} + C_2 e^{-2t}$.

18. The general solution is $y = C_1 \cos 10t + C_2 \sin 10t$.

19. A particular solution is $y = 7e^{3t} - e^{-2t}$.

20. A particular solution is $y = 12 e^{\pi - t} \sin 2t$.

21. A particular solution is $y = -2\pi t e^{-5t}$.

22. Consider all the nonzero solutions of the equation $y'' + 12y' + 36y = 0$, determine their behavior as $t \to \infty$.

23. Consider all the nonzero solutions of the equation $y'' - 2y' + 10y = 0$, determine their behavior as $t \to \infty$.

Answers B-1.3:

2. $y = C_1 e^{-t} + C_2 e^{-8t}$, $\quad y = \frac{15}{7} e^{-t} - \frac{1}{7} e^{-8t}$, 0

3. $y = C_1 e^{3t} \cos 4t + C_2 e^{3t} \sin 4t$, $\quad y = 2e^{3t} \cos 4t - \frac{7}{4} e^{3t} \sin 4t$, none

4. $y = C_1 e^{2t} + C_2 e^{4t}$, $\quad y = \frac{9}{2} e^{2t} - \frac{5}{2} e^{4t}$, $-\infty$

5. $y = C_1 e^{t/2} + C_2 e^{-3t}$, $\quad y = \frac{10}{7} e^{t/2} + \frac{4}{7} e^{-3t}$, ∞

6. $y = C_1 e^{4t} + C_2 t e^{4t}$, $\quad y = 2e^{4t} - 9te^{4t}$, $-\infty$

7. $y = C_1 e^{-2t} \cos 3t + C_2 e^{-2t} \sin 3t$, $\quad y = 2e^{-2t} \cos 3t + e^{-2t} \sin 3t$, 0

8. $y = C_1 e^{t/2} + C_2$, $\quad y = -2e^{t/2} + 4$, $-\infty$

9. $y = C_1 e^{-t/2} + C_2 e^{-2t}$, $\quad y = 2e^{-t/2}$, 0

10. $y = C_1 e^{t/4} + C_2 t e^{t/4}$, $\quad y = 2e^{t/4} - \frac{3}{2} t e^{t/4}$, $-\infty$

11. $y = 6e^{-2(t-5\pi)} - 2e^{-7(t-5\pi)}$

12. $y = 2e^{4(t+2)} - 9(t+2)e^{4(t+2)} = -16e^{4t+8} - 9te^{4t+8}$

13. $y = -2\cos\frac{t}{3} + 6\sin\frac{t}{3}$

14. $y = 5e^{-3(t-10)} \cos 5(t-10) + 2e^{-3(t-10)} \sin 5(t-10)$

15. $y = 2e^{t/2} - 10e^{t/5}$

16. $y'' - 2y' + y = 0$

17. $y'' - 3y' - 10y = 0$

18. $y'' + 100y = 0$

19. $y'' - y' - 6y = 0$

20. $y'' + 2y' + 5y = 0$

21. $y'' + 10y' + 25y = 0$

22. The solutions are of the form $y = C_1 e^{-6t} + C_2 t e^{-6t}$, they all approach 0 as $t \to \infty$.

23. The solutions are of the form $y = C_1 e^t \cos 3t + C_2 e^t \sin 3t$. The zero solution (i.e. when $C_1 = C_2 = 0$) approaches 0, all the nonzero solutions oscillate with an increasing amplitude and do not reach a limit.

Reduction of Order

Problem: Given a second order, homogeneous, linear differential equation (with non-constant coefficients) and a known nonzero solution y_1, find the general solution of the given equation.

To start, assume that there exists a second solution in the form of $y_2 = y_1\, v(t)$, for some differentiable function $v(t)$.

First we want to make sure the equation is written in the standard form with leading coefficient 1:

$$y'' + p(t)y' + q(t)y = 0.$$

Next, we will compute the Wronskian $W(y_1, y_2)(t)$ two different ways, using the two methods that we know. By the definition of Wronskian:

$$W(y_1, y_2) = \det\begin{pmatrix} y_1 & y_1 v(t) \\ y_1' & y_1' v(t) + y_1 v'(t) \end{pmatrix} = y_1 y_1' v(t) + y_1^{\,2} v'(t) - y_1 y_1' v(t) = y_1^{\,2} v'(t)$$

By the Abel's Theorem:

$$W(y_1, y_2) = C\, e^{-\int p(t)\,dt}, \qquad \text{where } C \neq 0.$$

The fact that $C \neq 0$ is important, because it guarantees the linear independence of y_1 and y_2.

The two expressions computed above are the Wronskian of the same two functions, therefore, the two expressions must be the same. Equate them:

$$y_1^{\,2} v'(t) = C\,e^{-\int p(t)\,dt}.$$

Therefore, $$v'(t) = C\,\frac{e^{-\int p(t)\,dt}}{y_1^{\,2}}, \qquad C \neq 0.$$

Integrate the right-hand side to find $v(t)$. Choose any convenient nonzero value for C. Letting $C = 1$ would work nicely, although it may not be the most convenient choice. Then find $y_2 = y_1\,v(t)$.

The general solution is still, of course, in the form $y = C_1 y_1 + C_2 y_2$. Therefore,

$$y = C_1 y_1 + C_2 y_2 = C_1 y_1 + C_2\,y_1\,v(t).$$

Note: It is actually not necessary to assume that $y_2 = y_1\,v(t)$. Although doing so makes the resulting first order differential equation easier to solve.

Example: If it is known that $y_1 = t$ is a solution of

$$t^2 y'' - t(t+2)y' + (t+2)y = 0, \qquad t > 0.$$

Find its general solution.

Rewrite the equation into the standard form

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 0$$

Identify $p(t) = -\dfrac{t+2}{t}$. Let $y_2 = y_1 v(t) = t\,v(t)$.

$$W(y_1, y_2) = \det\begin{pmatrix} t & tv(t) \\ 1 & v(t) + tv'(t) \end{pmatrix} = tv(t) + t^2 v'(t) - tv(t) = t^2 v'(t),$$

and,

$$W(y_1, y_2) = C\,e^{\int \frac{t+2}{t}dt} = C\,e^{\int\left(1+\frac{2}{t}\right)dt} = C\,e^{t+\ln(t^2)} = C\,t^2 e^t,$$

where $C \neq 0$.

Equating both parts: $\qquad t^2 v' = C\,t^2 e^t$

$$v' = C\,e^t \qquad \rightarrow \qquad v = C\,e^t + C_1$$

Choose $C = 1$ and $C_1 = 0$ $\rightarrow$ $v = e^t$. Therefore, $y_2 = y_1 v(t) = t\,e^t$.

The general solution is

$$y = C_1 y_1 + C_2 y_2 = C_1 t + C_2 t\,e^t.$$

Example: Find the general solution of the equation below, given that $y_1 = t^2 \cos(\ln t)$ is a known solution.

$$t^2 y'' - 3t y' + 5y = 0, \qquad t > 0.$$

Rewrite the equation into the standard form

$$y'' - \frac{3}{t} y' + \frac{5}{t^2} y = 0$$

Identify $p(t) = -\dfrac{3}{t}$. Let $y_2 = y_1\, v(t) = t^2 \cos(\ln t)\, v(t)$.

$$W(y_1, y_2) = t^4 \cos^2(\ln t) v'(t),$$

and,

$$W(y_1, y_2) = C\, e^{\int \frac{3}{t} dt} = C\, e^{3\ln(t)} = Ce^{\ln(t^3)} = C\, t^3, \quad C \neq 0.$$

Equating both parts: $\qquad t^4 \cos^2(\ln t)\, v' = C\, t^3$

$$v' = \frac{C}{t \cos^2(\ln t)} = \frac{C \sec^2(\ln t)}{t}$$

$$v = C \int \frac{\sec^2(\ln t)}{t}\, dt = C \tan(\ln t) + C_1 = C \frac{\sin(\ln t)}{\cos(\ln t)} + C_1$$

Choose $C = 1$ and $C_1 = 0 \;\rightarrow\; v = \tan(\ln t)$.

$$y_2 = y_1\, v(t) = t^2 \cos(\ln t) \tan(\ln t) = t^2 \sin(\ln t).$$

The general solution is

$$y = C_1 y_1 + C_2 y_2 = C_1 t^2 \cos(\ln t) + C_2 t^2 \sin(\ln t).$$

Exercises B-1.4:

1 – 7 For each equation below, a known solution is given. Find a second, linearly independent solution of the equation, and find the general solution.

1. $t^2y'' - 2ty' + 2y = 0,\quad t > 0,\quad y_1 = t^2.$

2. $t^2y'' - ty' - 3y = 0,\quad t > 0,\quad y_1 = t^{-1}.$

3. $ty'' + y' = 0,\quad t > 0,\quad y_1 = 1.$

4. $t^2y'' - 5ty' + 8y = 0,\quad t > 0,\quad y_1 = t^4.$

5. $t^2y'' - ty' + 10y = 0,\quad t > 0,\quad y_1 = t\sin(3\ln t).$

6. $(t-5)^2y'' - 2(t-5)y' + 2y = 0,\quad t > 5,\quad y_1 = (t-5)^2.$

7. $(t+2)^2y'' + 3(t+2)y' + y = 0,\quad t > -2,\quad y_1 = (t+2)^{-1}.$

8. Solve the initial value problem

$$t^2y'' - 3ty' + 4y = 0,\quad t > 0,\quad y(1) = -2,\quad y'(1) = 1.$$

Given that $y_1 = t^2\ln t$ is a known solution.

9. (a) Find the general solution of $t^2y'' - 2y = 0$, $t > 0$, given $y_1 = t^2$.
(b) Find the particular solution satisfying $y(1) = 6$ and $y'(1) = 9$.
(c) Show that the initial value problem $t^2y'' - 2y = 0$, $y(0) = 0$ and $y'(0) = 0$, does not have a unique solution by verifying that any of the infinitely many functions of the form $y = Ct^2$ is a solution, regardless of the value of C. Does this fact violate the Existence and Uniqueness Theorem?

Answers B-1.4:

1. $y = C_1 t^2 + C_2 t$
2. $y = C_1 t^{-1} + C_2 t^3$
3. $y = C_1 + C_2 \ln t$
4. $y = C_1 t^4 + C_2 t^2$
5. $y = C_1 t \sin(3 \ln t) + C_2 t \cos(3 \ln t)$
6. $y = C_1(t-5)^2 + C_2 (t-5)$
7. $y = C_1 \dfrac{1}{t+2} + C_2 \dfrac{\ln(t+2)}{t+2}$
8. $y = 5t^2 \ln t - 2t^2$
9. (a) $y = C_1 t^2 + C_2 t^{-1}$, (b) $y = 5t^2 + t^{-1}$

(Optional topic) Euler Equations

A second order *Euler equation* (also known as an *Euler-Cauchy equation*) is a second order homogeneous linear equation of the form

$$t^2 y'' + \alpha t y' + \beta y = 0, \qquad (**)$$

or, in standard form

$$y'' + \frac{\alpha}{t} y' + \frac{\beta}{t^2} y = 0 .$$

In a course at this level, variations of the Euler equation most frequently appear as examples and exercises in lectures about reduction of order. In this context we have seen a few of them in the previous section. This type of equations, however, is very interesting in its own right. Despite the non-constant nature of their coefficients, Euler equations can be easily solved in a way that is analogous to the characteristic equation method of solving constant coefficient homogeneous linear equations. We shall develop this solution technique for Euler equations in this section.

By visual inspection (or by peeking back at the exercises previously encountered in the section about reduction of order technique) we might deduce that a function of the form $y = t^r$ could be a solution of (**), $t \neq 0$. Therefore, similar to how we have previously derived the characteristic equation method, we will assume that, for some power, r, yet to be determined, there exists a solution $y = t^r$. We then substitute it into (**) to get a better idea about what r should be.

For the time being, let us consider only the case of $t > 0$. Start with the trial solution $y = t^r$, then $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Plug them into (**):

$$r(r-1)t^{r-2+2} + \alpha r t^{r-1+1} + \beta t^r = 0,$$

$$(r^2 - r + \alpha r + \beta)\, t^r = 0,$$

$$(r^2 + (\alpha - 1)r + \beta)\, t^r = 0.$$

Since $t^r \neq 0$, it follows that $r^2 + (\alpha - 1)r + \beta = 0$. This quadratic equation is the "characteristic equation" of (**). Whatever value of r, real or complex, that satisfies the characteristic equation will yield a nontrivial solution of (**) in the form $y = t^r$.

As you might have suspected, depending on the number and type of roots r of the characteristic equation, the equation (**) will have different forms of (real-valued) general solution. We will look at each case in turn.

Case I: There are two distinct real roots r_1 and r_2.

In this case $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$ are two solutions linearly independent everywhere on the interval $(0, \infty)$. (Exercise: check that their Wronskian is nonzero for $t \neq 0$.) Therefore, a general solution of (**) is

$$\boxed{y = C_1 y_1 + C_2 y_2 = C_1 t^{r_1} + C_2 t^{r_2}.}$$

Case II: There are two complex conjugate roots $r = \lambda \pm \mu i, \quad \mu > 0.$

In this case $y_1 = t^{r_1}$ and $y_2 = t^{r_2}$ remain two solutions linearly independent everywhere on the interval $(0, \infty)$. They are complex-valued functions, however:

$$y_1 = t^{\lambda + \mu i} = t^{\lambda} t^{\mu i} = t^{\lambda} e^{\mu(\ln t) i} = t^{\lambda}(\cos(\mu \ln t) + i \sin(\mu \ln t))$$
$$y_2 = t^{\lambda - \mu i} = t^{\lambda} t^{-\mu i} = t^{\lambda} e^{-\mu(\ln t) i} = t^{\lambda}(\cos(\mu \ln t) - i \sin(\mu \ln t))$$

In the same fashion as we have done for the constant coefficients second order linear equation earlier, we can produce the following pair of real-valued, linearly independent solutions using linear combinations.

$$u = (y_1 + y_2)/2 = t^{\lambda}\cos(\mu\ln t\,)$$

$$v = (y_1 - y_2)/2i = t^{\lambda}\sin(\mu\ln t\,)$$

Therefore, a real-valued general solution is

$$\boxed{y = C_1 t^{\lambda}\cos(\mu\ln t\,) + C_2 t^{\lambda}\sin(\mu\ln t).}$$

Case III: There is a repeated real root r.

Initially, we have only $y_1 = t^r$ as a solution. The second solution can be readily found by the method of reduction of order to be $y_2 = t^r \ln t$.

To wit: If $r = k$ is a repeated root, then the characteristic equation has coefficients $\alpha - 1 = -2k$, i.e., $p(t) = \dfrac{\alpha}{t} = \dfrac{-2k+1}{t}$; and $\beta = k^2$. Now, let $y_1 = t^k$ and $y_2 = t^k v$.

It follows that $W(y_1, y_2) = y_1^{\,2} v'(t) = t^{2k} v'(t)$, and by Abel's theorem, it is also $W(y_1, y_2) = C\, e^{\int \frac{2k-1}{t}dt} = C\, e^{(2k-1)\ln(t)} = C\, t^{2k-1}$, $\qquad C \neq 0.$

Hence, $t^{2k} v'(t) = C\, t^{2k-1} \qquad \rightarrow \qquad v'(t) = \dfrac{C}{t}, \qquad C \neq 0.$

Integrate to obtain $v(t) = C\ln t + C_1$, then set $C = 1$ and $C_1 = 0$.

We have $v(t) = \ln t$.

Consequently, $y_1 = t^k = t^r$, and $y_2 = y_1 \ln t = t^r \ln t$, are the two required fundamental solutions.

Therefore, a general solution is

$$\boxed{y = C_1 t^r + C_2 t^r \ln t\,.}$$

For $t < 0$, the general solution will take the same forms described above, except each formula will be in terms of $|\, t \,|$.

Example: Find the general solution of

$$t^2 y'' - 3t y' + 20 y = 0, \qquad t > 0.$$

The characteristic equation is $r^2 - 4r + 20 = 0$, which has roots $r = 2 \pm 4i$. Therefore, the general solution is

$$y = C_1\, t^2 \cos(4 \ln t\,) + C_2\, t^2 \sin(4 \ln t).$$

Example: Find the general solution of

$$t^2 y'' + 7t y' + 9 y = 0, \qquad t < 0.$$

The characteristic equation is $r^2 + 6r + 9 = (r + 3)^2 = 0$, which has a repeated root $r = -3$. Therefore, with $t < 0$, the general solution is

$$y = C_1 |\, t \,|^{-3} + C_2 |\, t \,|^{-3} \ln |\, t \,|.$$

Solution by substitution

Alternatively, the Euler equation can also be solved by a simple substitution. This approach seeks to convert an Euler equation into one with constant coefficients, thus establish a direct relation between their characteristic equations discussed previously.

Define $t = e^x$, thus $x = \ln t$, for $t > 0$. (Similarly, let $|t| = e^x$, thus $x = \ln |t|$, for $t < 0$.)

It follows that $\dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \dfrac{dy}{dt}e^x = \dfrac{dy}{dt}t$, and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}t\right) = \frac{d^2y}{dt^2}\frac{dt}{dx}t + \frac{dy}{dt}\frac{dt}{dx} = \frac{d^2y}{dt^2}t^2 + \frac{dy}{dt}t.$$

That is,

$$t\frac{dy}{dt} = \frac{dy}{dx}, \text{ and}$$

$$t^2\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} - \frac{dy}{dt}t = \frac{d^2y}{dx^2} - \frac{dy}{dx}.$$

Therefore, in terms of x, equation (**) becomes

$$\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) + \alpha\frac{dy}{dx} + \beta\, y = 0.$$

Or,

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta\, y = 0.$$

The equation now has constant coefficients, which can be solved using its characteristic equation $r^2 + (\alpha - 1)r + \beta = 0$.

Depending on the number and type of the roots of the characteristic equation, we have:

Case I: There are two distinct real roots r_1 and r_2.

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x} = C_1 e^{r_1 \ln t} + C_2 e^{r_2 \ln t} = C_1 t^{r_1} + C_2 t^{r_2} .$$

Case II: There are two complex conjugate roots $r = \lambda \pm \mu i, \qquad \mu > 0$.

$$\begin{aligned} y &= C_1 e^{\lambda x} \cos \mu x + C_2 e^{\lambda x} \sin \mu x \\ &= C_1 e^{\lambda \ln t} \cos(\mu \ln t) + C_2 e^{\lambda \ln t} \sin(\mu \ln t) \\ &= C_1 t^{\lambda} \cos(\mu \ln t) + C_2 t^{\lambda} \sin(\mu \ln t). \end{aligned}$$

Case III: There is a repeated real root r.

$$y = C_1 e^{rx} + C_2 x e^{rx} = C_1 t^{r} + C_2 t^{r} \ln t .$$

As can be seen, the two methods arrive at the identical results.

Translated Euler Equations

Euler equations can also be written in terms of $(t - \delta)$, for a constant δ, instead of t. Namely,

$$(t - \delta)^2 y'' + \alpha (t - \delta) y' + \beta y = 0.$$

In such a case, simply substitute $T = t - \delta$. This allows us to solve it the same way previously discussed, but in terms of T. When $t < \delta$, then $T < 0$, the solution will have to be expressed in terms of $|\, T \,| = |\, t - \delta \,|$, as usual.

Example: Find the general solution of

$$(t+6)^2 y'' + 2(t+6)y' - 12y = 0, \qquad t > -6.$$

The characteristic equation is $r^2 + r - 12 = 0$, which has roots $r = 3$ and -4. Therefore, the general solution is

$$y = C_1 (t+6)^3 + C_2 (t+6)^{-4}.$$

Example: Find the general solution of

$$(t-\pi)^2 y'' - 9(t-\pi) y' + 25 y = 0, \qquad t < \pi.$$

The characteristic equation is $r^2 - 10r + 25 = 0$, which has a repeated root $r = 5$. Therefore, with $T = t - \pi < 0$, the general solution is

$$y = C_1 | t - \pi |^5 + C_2 | t - \pi |^5 \ln | t - \pi |.$$

Summary

Given a second order Euler equation

$$t^2 y'' + \alpha t y' + \beta y = 0, \qquad t > 0.$$

Solve its characteristic equation $r^2 + (\alpha - 1)r + \beta = 0$. The general solution depends on the type of roots obtained:

1. When there are two distinct real roots r_1, r_2.

$$y = C_1 t^{r_1} + C_2 t^{r_2}.$$

2. When there are two complex conjugate roots $r = \lambda \pm \mu i$.

$$y = C_1 t^{\lambda} \cos(\mu \ln t) + C_2 t^{\lambda} \sin(\mu \ln t).$$

3. When there is one repeated real root r.

$$y = C_1 t^{r} + C_2 t^{r} \ln t.$$

With $t = 0$ being the only discontinuity of $p(t)$ and $q(t)$, when $t_0 > 0$, in each case above there is always a unique solution valid everywhere on $(0, \infty)$ for any pair of initial conditions $y(t_0) = y_0$ and $y'(t_0) = y'_0$. When $t_0 < 0$, replace every t in each formula above by $|t|$, and a unique solution valid everywhere on $(-\infty, 0)$ can always be found.