## 1. Introduction

Random matrices were first introduced in physics by Wigner in the 1950's as a tool to study the statistical properties of resonances observed in the scattering of neutrons by heavy nuclei (see [1–4] and other articles reprinted in ref. [5]). The ideas involved are very general, as we shall see in the presentation that follows, and have been applied successfully to a variety of quantum-mechanical systems: some of these applications will be described here and also in other courses of this School; an application to chaotic dynamics can be found, for instance, in ref. [6]. It is this *universal* description that makes the topic so appealing. Whenever a universal behaviour has been discovered in physics, its understanding has represented a great theoretical challenge: let us just cite, as a beautiful example, the universality of critical phenomena [7], whose comprehension in the past has been of far reaching consequences.

In his original statistical analysis of nuclear resonances, Wigner approached the problem from the standpoint of scattering theory [1–3]. For example, it was in those early papers that the notion of a statistical $R$ function was introduced. Since the construction of the $R$-function was conceived in terms of its poles and residues (associated with bound states inside a box), the attention of a number of people concentrated on the statistical properties of the discrete spectrum of bound systems from a very general point of view. This is in fact the topic of a large part of the material contained in the reviews of refs. [5,8,9] (where applications to a number of physical systems can already be found) and forms the subject matter of section 2 of the present course.

The statistical analysis of the original scattering problem was undertaken by several groups (see, e.g., the papers referred to in chapter VIII of ref. [8] and in section 3 of this course), not always from the same point of view: sometimes a phenomenological approach was taken; in some papers the description was done in terms of the statistics of the poles and residues of the scattering matrix $S$, or those of the $R$, or $K$ matrix; in others, statistics on the $S$ matrix itself was done. Again, some of the ideas involved are very general and have been applied to a variety of physical systems, whose dimensions may differ by many orders of magnitude - atomic nuclei, mesoscopic systems, microwave cavities! This forms the topic of section 3 of these notes. We first consider those systems in which the scattering process can be described in terms of an "equilibrated, delayed"

component only, and then those that also need a "prompt, direct" component, for their understanding. Finally, we present some random-matrix models intended to describe the scattering produced by disordered systems, where the time scale set by a diffusive process has a dominant role.

A number of appendices are included for the sake of completeness. Appendix A gives, in its first part, a brief survey of the quantum-mechanical scattering theories introduced by various authors; in its second part it explains, by means of simple examples, some of the basic notions of scattering theory used in section 3: the scattering matrix $S$ and its properties (unitarity and its analytic structure in the complex energy plane), the transfer matrix, its properties and its relation to the $S$ matrix. A central-limit theorem of a novel kind, relevant to 1-channel scattering theory, is verified, for a particular situation, in Appendix B. In appendix C a Fokker–Planck equation relevant to the description of scattering in the presence of diffusion is obtained.

In the presentation that follows we deal with situations where the complexity of the problem washes out most of the details and leads to a behaviour that reflects the *symmetries* of the system, depends only on a rather limited number of *relevant parameters* (generally of a macroscopic nature, having a clear physical significance) and is *insensitive to other details* of the problem. In a number of cases one has discovered a generalized *central-limit theorem* (CLT) responsible for this behaviour. In these situations, a microscopic calculation might end up being a "scaffolding", due to the final insensitivity to most of the details. It is then conceptually appealing to construct directly the statistical distribution of the quantities of interest by imposing the symmetry constraints and the values of the relevant parameters alluded to above. This procedure may determine the distribution uniquely; if it does not, it has been frequently found advantageous to make a selection on the basis of a *maximum-entropy* criterion. Such a criterion allows one to pick the "most probable" distribution among those that satisfy the given constraints; alternatively, one could say that it treats the statistical variables of the problem "as randomly as it is allowed" by those constraints. In a sense, the above philosophy is best expressed in William of Occam's (1300–1349) famous dictum:

*"Essentia non sunt multiplicanda praeter necessitatem",*

known as the "Occam Razor". Occam's statement, which, literally, means: "Entities do not have to be multiplied beyond necessity", was rephrased by Bertrand Russel [10] as: "If in a certain science everything can be interpreted without a certain hypothesis, there is no reason to use it".



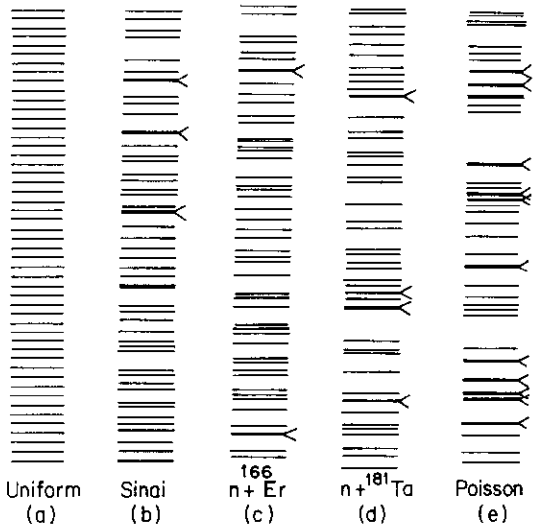


Fig. 1. Illustration of the statistical regularities that occur in a variety of situations, as discussed in the text.

## 2. Random matrices and spectral statistics

The energy spectra of complex quantum-mechanical systems are, in general, not amenable to a detailed level-by-level analysis. In many-body systems one finds complicated states that no one even knows how to calculate; in chaotic single-particle systems, even when one can solve numerically the Schrödinger equation, one does not learn much with an individual-level analysis. In contrast, a statistical description like the one to be discussed below, besides being, in many cases, the only feasible one, reveals features which would otherwise remain hidden.

As an illustration of the *statistical regularities* that occur in a surprising variety of situations, consider the spectra shown in fig. 1 [8,11]. Each contains 50 levels and they have all been rescaled to the same spectrum span. Fig. 1a shows a spectrum of equally-spaced levels (picket fence); Fig. 1b shows the eigenvalues of a Sinai billiard for a *fixed exact symmetry* (reflection symmetry); (c), (d) are experimental data for neutron resonances: in the first all the levels have the same exact symmetry ($J^\pi = 1/2^+$), while the second one contains mixed exact symmetries ($J^\pi = 3^+, 4^+$); finally, fig.1e shows a Poisson's sequence, corresponding to levels thrown at random. The "arrowheads" mark the occurrence of pairs of adjacent levels with spacings smaller than one quarter of the average. We observe, from right to left, a decrease in the probability of small spacings, or the

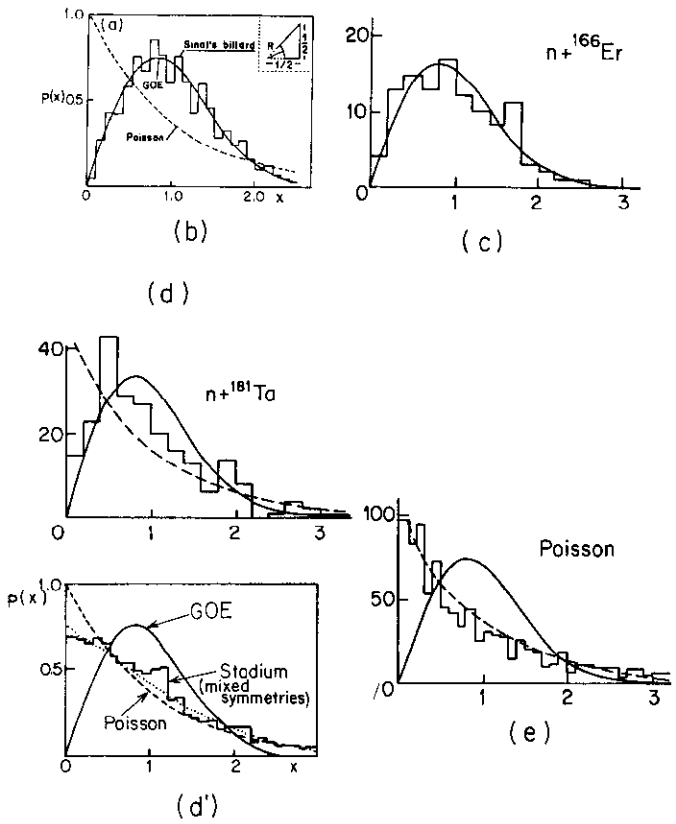


Fig. 2. Nearest-neighbour spacing distribution for spectra b-e of fig. 1; panel d′ corresponds to the spectrum of a stadium with the four possible symmetries included.

building up of a phenomenon known as *level repulsion*. This is seen more clearly in fig. 2, that gives a histogram of nearest-neighbour spacings for cases (b)–(c). The levels of fig. 2b, as well as those of fig. 2c, all corresponding to the same exact symmetry, are fitted very well by "Wigner's surmise" [12]

$$p(x) = (\pi x/2)\mathrm{e}^{-\pi x^2/4}. \tag{2.1}$$

Here, $x$ denotes a spacing measured in units of the mean spacing. Case (e) is well described by Poisson's distribution [see ref. [5], Introduction]

$$p(x) = \mathrm{e}^{-x}, \tag{2.2}$$



shown as a dashed line. On the other hand, spectrum (d), corresponding to mixed exact symmetries, is shown in the figure to have a spacing distribution somewhat intermediate between (2.1) and (2.2); a similar situation is observed in fig. 2d′, that shows the spacing distribution for the spectrum of a stadium, containing all levels corresponding to the four possible symmetries [6,11]. It is also apparent from fig. 1 that the spectra (b) and (c) deviate from the picket-fence (a) less strongly than (e) : to this *spectral rigidity* we shall come back later.

Figs. 1b, c and 2b, c are just one piece of evidence, with spectra arising from very different physical problems – a Sinai billiard and a nucleus – of a statistical regularity actually observed when the classically related problem is fully chaotic [6]: we shall say that these systems, that behave statistically in a similar way, belong to the same *universality class*.

The study of the universal behaviour occurring in complex spectra is our purpose in what follows. We shall not touch upon those features which appear to be system dependent, such as those related with the short orbits in a semiclassical description (see ref. [6], p. 156; see also ref. [13] for a discussion of energy level statistics in small metal samples). In the scattering problems studied in the next section, we also consider the universal properties first (section 3.1); we show next that in various cases of physical interest one can identify a number of system-dependent *relevant parameters* which, once specified, do determine the statistical scattering properties of the system.

For our theoretical analysis it is useful to treat a universality class through Wigner's notion of *ensemble of random matrices* [1–5,14], i.e., Hamiltonian matrices

$$H = \begin{pmatrix} H_{11} & \cdots & H_{1N} \\ \vdots & \ddots & \vdots \\ H_{N1} & \cdots & H_{NN} \end{pmatrix} \tag{2.3}$$

distributed according to a probability law. This is reminiscent of the notion of Gibb's ensemble (which is defined, though, for a fixed Hamiltonian) in statistical mechanics. Similar issues as in statistical mechanics then arise, like that of *ergodicity*: just as the question of the equality of time averages and ensemble averages is important in classical statistical mechanics, we have to face the question of spectral vs. ensemble averages in the present context.

Following the ideas outlined in the Introduction, we propose a statistical model for our random matrices on the basis of a maximum-entropy criterion. We thus make a digression to introduce this concept.

### 2.1. The concepts of information and entropy of a statistical distribution

Suppo. e we wish to accomodate $N$ objects in $k$ boxes. If $N_i$ denotes the number of particles in box $i$, the weight $W$ of the configuration $N_1, N_2, \ldots, N_k$ is [15]

$$W = \frac{N!}{N_1!N_2!\cdots N_k!}. \tag{2.4}$$

*In the absence of other physically relevent restrictions*, the *most probable configuration is the one of maximum weight* : $N_1 = N_2 = \cdots = N_k$. In classical statistical mechanics one finds the Maxwell–Boltzmann distribution using a maximum-weight argument. There, the objects are particles and the boxes represent equal volume elements in $\mu$-space, the energy $\epsilon_i$ being assigned to them; but one has further restrictions: in the microcanonical ensemble the total number of particles $N$ *and* the total energy $E$ are fixed; for $N \gg 1$, these (plus the total volume) are, for a thermodynamical description, *the only relevant physical quantities*, other dynamical details being irrelevant. For given $N, E$, the most probable configuration, i.e. the one with maximum weight, is given by the well known Maxwell–Boltzmann expression $N_i = \exp(-\alpha' - \beta\epsilon_i)$, $\alpha'$ and $\beta$ being Lagrange multipliers. This statement can be rephrased in terms of the *entropy* per particle $\mathcal{S} = N^{-1} \ln W$. Using Stirling's approximation, $\mathcal{S}$ can be written as

$$\mathcal{S} = -\sum_i p_i \ln p_i, \tag{2.5}$$

where $p_i = N_i/N$ is the fraction of particles (or probability to find a particle) in box $i$. With the restrictions

$$\sum_i p_i = 1, \tag{2.6}$$

$$\sum_i \epsilon_i p_i = \bar{\epsilon}, \tag{2.7}$$

$\mathcal{S}$ is maximum when

$$p_i = \mathrm{e}^{-\alpha - \beta\epsilon_i}. \tag{2.8}$$

The quantity $\mathcal{S}$, the thermodynamic entropy in the statistical mechanical problem, can also be assigned to an arbitrary distribution $p_i$ , $(i = 1, \ldots, k)$, in which case the term entropy is sometimes qualified as *Shannon entropy* or *information-theory entropy* [16–20]. The information is defined as $I = -\mathcal{S}$. The distribution (2.8) is the one with *minimum information*, or *maximum entropy*, consistent with the constraints (2.6), (2.7), which in turn represent the only *physically relevant quantities* in this problem.



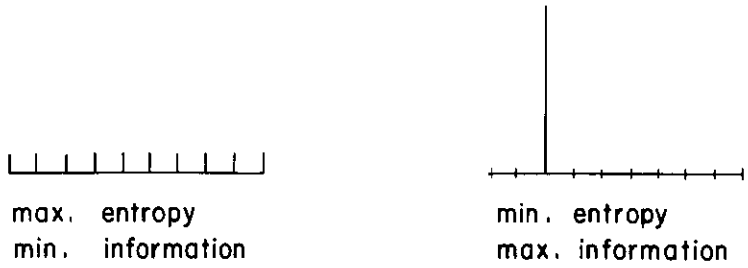


Fig. 3. A simple example of maximum and minimum-entropy distributions.

We contrast in fig. 3 the distribution of minimum information (or maximum entropy) consistent with the constraint (2.6) alone, with a distribution of maximum information (or minimum entropy); in the former case all the $p_i$'s, $i = 1, \ldots, k$ are equal to $1/k$ and $\mathcal{S} = \ln k$; in the latter case, $\mathcal{S} = 0$.

It is interesting that the form (2.5) for the entropy can also be found from a *uniqueness theorem* ([18], p. 9). Suppose we require the entropy $\mathcal{S}$, the negative of the information $I$, to have the following properties:

1. For given $k$ and for $\sum_{i=1}^{k} p_i = 1$, the function $\mathcal{S}(p_1, \ldots, p_k)$ takes its largest possible value for $p_i = 1/k$.

2. For two (in general statistically dependent) sets of events $\{A_k\}$ and $\{B_k\}$, the entropy of the combined set satisfies $\mathcal{S}(AB) = \mathcal{S}(A) + \sum_k p_k \mathcal{S}_k(B)$. Here, $\mathcal{S}_k(B)$ is the conditional entropy of the set $\{B_\ell\}$, calculated on the assumption that the event $A_k$ actually occurred.

3. Adding an impossible event does not change the entropy; $\mathcal{S}(p_1, \ldots, p_k, 0) = \mathcal{S}(p_1, \ldots, p_k)$.

Then, if $\mathcal{S}(p_1, \ldots, p_k)$ is continuous with respect to all its arguments and has the above three properties, it must have the form (2.5), up to a multiplicative constant.

Consider now an example with a continuous variable. Let $x \in (-\infty, +\infty)$. The probability density $p(x)$ is related to the differential probability by

$$\mathrm{d}P(x) = p(x)\mathrm{d}x, \tag{2.9}$$

and the entropy is defined as

$$\mathcal{S} = -\int p(x) \ln p(x) \mathrm{d}x. \tag{2.10}$$

From the above discussion it is clear that among all the distributions that are normalized and have fixed first and second moments (equal to $\mu$ and $\mu^2 + \sigma^2$,

respectively) the Gaussian

$$p(x) = \frac{\mathrm{e}^{-(x-\mu)^2/\sigma^2}}{(2\pi\sigma^2)^{1/2}} = \mathrm{e}^{-\alpha-\beta x-\gamma x^2} \tag{2.11}$$

has the largest entropy. Let us recall an interesting situation where a Gaussian distribution arises. Consider a number $n$ of statistically independent variables with a common *arbitrary* (the actual restrictions are very mild!) distribution. The distribution of their sum becomes approximately Gaussian for $n \gg 1$ and, as $n \to \infty$ and with a proper renormalization, it tends to a zero-centered Gaussian with unit variance. This result is known as the *central-limit theorem (CLT)* of statistics [21]. We thus see that the resulting distribution is sensitive only to the centroid and width of the original one, *all the other details being irrelevant*! This is precisely reflected in the fact that the information (entropy) carried by a Gaussian is smallest (largest) among those distributions that have the same centroid and width. This interpretation has to be contrasted with the more subjective one of Jaynes [17], who would look for a distribution having maximum entropy, "while agreeing with whatever information is *given*".

As another example of a continuous variable, consider the angle $\theta \in (0, 2\pi)$, with the relation

$$\mathrm{d}P(\theta) = p(\theta)\mathrm{d}\theta \tag{2.12}$$

between the differential probability and the probability density. The entropy is defined as

$$\mathcal{S} = -\int p(\theta)\ln p(\theta)\mathrm{d}\theta. \tag{2.13}$$

With no other constraint except normalization, $\mathcal{S}$ is maximum for $p(\theta) = 1/2\pi$. On the other hand, with the constraint $\langle\cos\theta\rangle = \mu$, $\mathcal{S}$ is maximum with

$$p(\theta) = \mathrm{e}^{-\alpha-\beta\cos\theta}. \tag{2.14}$$

We observe that, for a continuous variable, we have to specify how to factorize the differential probability into a probability density times what has sometimes been called the "prior" [20] ($\mathrm{d}x$ and $\mathrm{d}\theta$ in the above examples). The choice of the prior corresponds to making a definite postulate of "equal a priori probabilities", as, after all, one always does in statistical studies. For a compact space, the prior is normalizable and defines the maximum-entropy distribution in the absence of constraints. In many cases of interest there is a "natural" way to "measure" the quantities in question: let us make it clear, though, that it is the confrontation with experiment that finally decides whether the choice is a reasonable one or not. In closing, let us just remark that choosing the prior as the *invariant measure* under a certain *symmetry operation* has the appealing consequence that the resulting entropy is also invariant under that operation [22]. As an illustration, in the above example $\mathrm{d}\theta$ remains invariant under the addition of an arbitrary constant phase $\theta_0$; i.e.



$$\theta' = \theta + \theta_0, \quad \mathrm{d}\theta = \mathrm{d}\theta'. \tag{2.15}$$

Calling $p(\theta)$ and $q(\theta')$ the probability densities of the variables $\theta$ and $\theta'$, respectively, we have

$$p(\theta)\mathrm{d}\theta = q(\theta')\mathrm{d}\theta'; \tag{2.16}$$

since $\mathrm{d}\theta = \mathrm{d}\theta'$ (eq. (2.15)),

$$q(\theta') = p(\theta(\theta')) = p(\theta' - \theta_0). \tag{2.17}$$

For the entropy we then have

$$\mathcal{S}[p(\theta)] = -\int p(\theta)\ln p(\theta)\mathrm{d}\theta,$$

so that, applying eqs. (2.16) and (2.17), we find

$$\mathcal{S}[p(\theta)] = \int q(\theta')\ln q(\theta')\mathrm{d}\theta' = \mathcal{S}[q(\theta')].$$

### 2.2. *An ensemble of real symmetric matrices*

We first make a short digression. In a three-dimensional Euclidean space the standard *arc element* is given by

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{2.18}$$

and the *volume element* by

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,. \tag{2.19}$$

Notice that both quantities are invariant under *rotations* as well as under *translations*. Should we require that the volume element be invariant under rotations only, we would have the freedom to multiply (2.19) by an arbitrary nonnegative function $f(r)$ of the radial coordinate $r = (x^2+y^2+z^2)^{1/2}$; this function reduces to a constant if $dV$ has to remain invariant under translations as well.

In the language of the previous subsection, (2.19) is the prior in terms of which we write a differential probability in our three-dimensional Euclidean space; i.e.

$$\mathrm{d}P(x, y, z) = p(x, y, z)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,; \tag{2.20}$$

we refer to $p(x, y, z)$ as the probability density.

For future purposes we point out that the above are special cases of a more general situation studied in differential geometry; there one considers the arc element

$$\mathrm{d}s^2 = \sum g_{\mu\nu}(x)\delta x_\mu \delta x_\nu, \tag{2.21}$$

(written in terms of independent variables), through which the metric tensor $g_{\mu\nu}(x)$ is defined. Assuming that $ds^2$ remains invariant under the transformation $x_\mu = x_\mu(x'_1, x'_2, \ldots)$, one can prove that the volume element

$$\mathrm{d}V = |\det g(x)|^{1/2} \prod_\mu \mathrm{d}x_\mu \tag{2.22}$$

remains invariant under the same transformation [23].

We now go back to our random-matrix problem. Consider a system that is invariant under the operation of time-reversal; for integral spins, or for half-integral spins and rotational invariance, one can always find a basis in which the Hamiltonian matrix $H$ is real and symmetric ([5], Introduction and [9]). This is the case for the problems illustrated in Figs. 1b,c and 2b,c that were discussed at the beginning.

First we have to define a way to "measure" real symmetric matrices ([5], p.32 and [24]) of dimensionality $N$. In analogy with the above 3D example, eqs. (2.18)-(2.20), to the variation $\delta H$ we associate the "arc element"

$$\mathrm{d}s^2 = \mathrm{tr}(\delta H)^2 \tag{2.23}$$

$$= \sum_i (\delta H_{ii})^2 + 2\sum_{i<j} (\delta H_{ij})^2, \tag{2.24}$$

(in terms of independent variables) and the volume element (up to a multiplicative constant)

$$\mathrm{d}\mu(H) = \prod_{i<j} \delta H_{ij} \prod_i \delta H_{ii}. \tag{2.25}$$

Just as in the example, the arc element (2.23) and the measure (2.25) remain invariant under a real orthogonal transformation (a change of basis in Hilbert space that maps the space of real symmetric matrices unto itself) as well as under the addition of a constant matrix. Using the above measure we then write the differential probability $\mathrm{d}P$ as

$$\mathrm{d}P(\{H_{ij}\}) = p(\{H_{ij}\})\mathrm{d}\mu(H); \tag{2.26}$$

we refer to $p(\{H_{ij}\})$ as the probability density.



In the cases discussed at the beginning of this section in connection with figs. 1 and 2, as well as in various numerical simulations and experimental situations, one finds a clear-cut separation between the properties of fluctuations and the secular variation of the level density: while the former have the universal character that we discussed at the beginning of this section, the latter is, generally speaking, system dependent [8,22,25]. It is then appropriate to look for an ensemble that has minimum information (or maximum entropy) among all those that have a prescribed density [22]. This task will be undertaken in subsection 2.4. At the present moment we adopt a simpler point view, that leads to an ensemble that has been analytically studied in great detail [9]. Suppose we agree to describe those situations in which, once the energy scale is specified, the *statistical properties of the spectrum are insensitive to most details*. An ensemble that reflects these properties is one of minimum information (or maximum entropy) once *the average strength of the Hamiltonian matrices*

$$\langle \mathrm{tr}(H^2) \rangle = \left\langle \sum_i (H_{ii})^2 + 2\sum_{i<j} (H_{ij})^2 \right\rangle \tag{2.27}$$

*is specified* [22]; the answer is

$$p(\{H_{ij}\}) = \mathrm{e}^{-\lambda - \mu \mathrm{tr}(H^2)}, \tag{2.28}$$

where $\lambda$ and $\mu$ are Lagrange multipliers. Eq. (2.28) defines the *Gaussian Orthogonal Ensemble* (GOE): $p(H)$ depends on $H$ only through a trace and is thus *invariant under orthogonal transformations*; in addition, except for the condition of symmetry, the various matrix elements are statistically independent Gaussian variables.

As a simple illustration, consider the case $N = 2$ (ref. [5], p. 53). We can write $p(H)$ as

$$p(H_{11}, H_{12}, H_{22}) = \mathrm{e}^{-\lambda - \mu(H_{11}^2 + H_{22}^2 + 2H_{12}^2)}. \tag{2.29}$$

To find the spectral properties, we make a change of variables to the eigenvalues $E_1, E_2$ and the angle $\theta$ of the orthogonal transformation that diagonalizes $H$; the measure (2.25) can then be expressed as

$$\mathrm{d}\mu(H) = |E_1 - E_2| \mathrm{d}E_1 \mathrm{d}E_2 \mathrm{d}\theta. \tag{2.30}$$

As a consequence, the joint probability density of the eigenvalues is

$$w(E_1, E_2) = |E_1 - E_2| e^{-\lambda - \mu(E_1^2 + E_2^2)}. \tag{2.31}$$

We first observe that the Jacobian $|E_1 - E_2|$ vanishes when the two eigenvalues coincide, thus giving rise to the level repulsion that was described at the beginning

Table 1

| $[H,\theta]$ | Spin | $[H,J]$ | Ensemble | $\beta$ |
|---|---|---|---|---|
| 0 | integer | | orthogonal | 1 |
| 0 | 1/2 integer | 0 | orthogonal | 1 |
| 0 | 1/2 integer | $\neq 0$ | symplectic | 4 |
| $\neq 0$ | | | unitary | 2 |

of this Section. If we go one step further and express (2.31) in terms of "center of mass" and "relative coordinates", we find precisely Wigner's spacing distribution (2.1)! We only mention here that the spacing distribution has also been calculated in the $N \to \infty$ limit, with the remarkable result that the Wigner surmise (2.1) is an excellent approximation to it [9].

### 2.3. *Universality classes*

The orthogonal case studied above belongs to one of the three universality classes arising from *general symmetry arguments*. We have the scheme shown in the table [9,26], where $J$ denotes the angular momentum operator and $\theta$ the time-reversal operator (see [5], Introduction; [27,28]). Time-reversal invariance, as is realized in the absence of a magnetic field, implies $[H,\theta] = 0$. We have already encountered the *orthogonal* case in the above presentation. In the *unitary* ensemble the Hamiltonian matrices are Hermitean and are transformed among themselves (an automorphism) under unitary transformations. In the *symplectic* case the automorphism is induced by a subgroup of the unitary matrices, called symplectic. Finally, the last column of the table assigns, to each one of the three universality classes, a parameter $\beta$, that counts the number of real parameters associated with each matrix site: 1 for real matrices, 2 for complex ones and, in the language of quaternions, 4 for matrices whose elements are linear combinations, with real coefficients, of the 4 quaternion units. As we shall see in subsection 2.5, $\beta$ plays, formally, a role similar to the inverse temperature in statistical mechanics.

### 2.4. *The Gaussian Ensembles (GE)*

If we maximize the entropy, constrained, as we did above, by a specified average strength of the Hamiltonian matrices, we find the *Gaussian Orthogonal, Unitary and Symplectic Ensembles (GOE, GUE, GSE)*, for $\beta = 1, 2, 4$, respectively. The



resulting joint probability density of eigenvalues is given by

$$w(E_1, \ldots, E_N) = C \prod_{i<j} |E_i - E_j|^\beta \prod_k e^{-\mu E_i^2}. \tag{2.32}$$

Eq. (2.31) above is a particular case of (2.32) for $\beta = 1, N = 2$. We summarize below some of the results that are found [5,6,8,9] from the joint distribution of eigenvalues, Eq. (2.32).

1. *The density of eigenvalues.* The "microscopic density" is defined as

$$\widehat{\rho}(E) = \sum_{i=1}^{N} \delta(E - E_i) \tag{2.33}$$

and the mean level density, or simply the density (a *one-point function*), is then given by

$$\rho(E) = \langle \widehat{\rho}(E) \rangle . \tag{2.34}$$

In terms of the rescaled energy $x = E/(\beta v^2 N)^{1/2}$ one finds, for $\rho(E)$, the so-called "semicircle"

$$\rho(x) = (N/2\pi)(4 - x^2)^{1/2}. \tag{2.35}$$

2. A *short range repulsion* is found: the nearest-neighbour spacing distribution behaves as $x^\beta$ when $x \to 0$.

3. *Long-range rigidity.* We first encountered this concept in our discussion of fig. 1. To be more precise, we present in fig. 4 a small segment of a 1206-dimensional spectrum resulting from a complex many-body calculation [8]. The left part shows the actual energy levels, while the right part shows an equally-spaced spectrum with the same local density, that actually originates from the bottom of the spectrum and is computed by an appropriate smoothing of the actual density. We find the surprising result that the actual level deviations are of the order of a single spacing unit! One can compute the fluctuation of a single energy level for the above ensembles. In units of the mean local spacing $\Delta$ one finds, in the center of the spectrum ( [8,9]; see ref. [29] for the circular ensemble discussed in the next section)

$$\langle (\delta x)^2 \rangle \approx \frac{\ln 2N}{\beta \pi^2}. \tag{2.36}$$

For $N = 10^{12}$, the level motion is only about 1.7 spacings!

The *number variance* $\Sigma^2(r)$, i.e. the variance of the number of levels $n$ in an interval $\delta E = r\Delta$, is also found to have a logarithmic dependence [8,9,30]; for

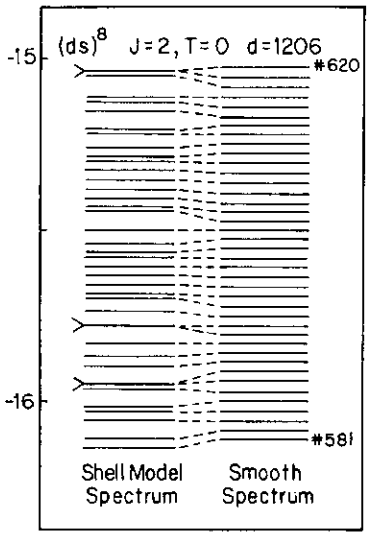


Fig. 4. Illustration of the spectral rigidity.

example, for $\beta = 2$ one finds

$$\Sigma^2(r) = \frac{1}{\pi^2}[\ln(2\pi r) + \gamma + 1] + O(\frac{1}{\pi^2 r}), \tag{2.37}$$

$\gamma$ being Euler's number.

In contrast, a Poisson's sequence of statistically independent levels gives rise to the much larger fluctuation

$$\Sigma^2(r) = r(1 - r/N) \overset{N\to\infty}{\longrightarrow} r. \tag{2.38}$$

This last result is shown by writing the number of levels inside $\delta E$ at $E$ as $n = \sum_{i=1}^{N} \theta_{E,\delta E}(E_i)$, where $\theta_{E,\delta E}(E_i)$ equals 1 if $E_i$ lies inside the interval $\delta E$ at $E$ and 0 otherwise; one then calculates var$(n)$ considering the levels to be statistically independent.

4. *Two-point functions.* In terms of the microscopic density (2.33) one defines the two-point function

$$\rho_2(E, E') = \langle \hat{\rho}(E)\hat{\rho}(E') \rangle - \langle \hat{\rho}(E) \rangle \langle \hat{\rho}(E') \rangle . \tag{2.39}$$

Separating out from the first term on the right-hand side of (2.39) the $i = j$ term arising from (2.33), one defines the *two-level cluster function* $Y_2(E, E')$ through



the equation

$$\frac{\rho_2(E, E')}{\rho(E)\rho(E')} = \frac{\delta(E - E')}{\rho(E)} - Y_2(E, E'). \tag{2.40}$$

One can also write

$$Y_2(E, E') = 1 - \frac{R_2(E, E')}{\rho(E)\rho(E')}, \tag{2.41}$$

where

$$R_2(E, E') = \sum_{i\neq j} \langle \delta(E - E_i)\delta(E' - E_j) \rangle \tag{2.42}$$

$$= N(N - 1)P_2(E, E') \tag{2.43}$$

is the relative frequency of pairs of levels and $P_2(E, E')$ the two-level joint probability density.

In a region of constant density, the right-hand side of (2.40) becomes $\delta(r) - Y_2(r)$, where $r = (E - E')/\Delta$ is the energy separation measured in units of the mean spacing $\Delta$.

For a Poisson spectrum consisting of $N$ levels inside an interval $L$, $P_2(E, E') = 1/L^2$ and $Y_2(r) = 1/N \to 0$ as $N \to \infty$. For a spectrum with level repulsion, as is the case for the matrix ensembles studied here, $P_2(E, E) = 0$ and $Y_2(0) = 1$; as $E - E' \to \infty$, the levels become independent and $Y_2(r) \to 0$. As an illustration, we cite the result for $\beta = 2$:

$$Y_2(r) = \left(\frac{\sin \pi r}{\pi r}\right)^2. \tag{2.44}$$

The equally-spaced peaks of $Y_2(r)$ indicate the presence of a *crystalline structure of long-range order*, again a manifestation of the spectral rigidity already noted above. For comparison, a picket-fence spectrum would give $Y_2(r) = 1 - \sum_{p\neq 0} \delta(r - p)$. These three cases are reminiscent of the behaviour of gas, liquid and solid systems in statistical mechanics.

The *2-level form factor* is defined as the Fourier transform of $Y_2(r)$ ; i.e.

$$b(k) = \int_{-\infty}^{\infty} Y_2(r) e^{2\pi i k r} dr. \tag{2.45}$$

For the $\beta = 2$ case of Eq. (2.44) one finds

$$\begin{aligned} b(k) &= 1 - |k|, \quad |k| < 1 \\ &= 0, \qquad |k| > 1. \end{aligned} \tag{2.46}$$

The physical significance of the form factor is particularly appealing. Suppose we construct a wave packet as a linear combination of $N$ states - contained in a stretch of energy $\delta E$, the mean spacing being $\Delta$- with identical coefficients $1/\sqrt{N}$. The overlap of the wave function at time $t$ with that at time 0, squared and ensemble averaged, tends to $1/N$ as $N \to \infty$, at a rate determined by $b(\omega t/2\pi)$, where $\omega = \Delta/\hbar$ (see also [31,32]).

5. A *linear statistic* $F$ is a function of the eigenvalues, with the form

$$F = \sum_i f(E_i) = \int f(E)\widehat{\rho}(E)\,\mathrm{d}E. \tag{2.47}$$

In (2.47) the microscopic density of Eq. (2.33) was used. The ensemble average and variance of $F$ can then be expressed in terms of the mean density $\rho$ and the two-point correlation function $\rho_2$ of eq. (2.39) as [34]

$$\langle F\rangle = \int f(E)\rho(E)\,dE \tag{2.48}$$

$$\mathrm{var}F = \int\int f(E)f(E')\rho_2(E,E')\mathrm{d}E\mathrm{d}E'. \tag{2.49}$$

Using (2.40) and (2.45) one finds, *in a region of constant density* (with mean spacing $\Delta$)

$$\mathrm{var}F = \frac{1}{\Delta}\int \phi(t)\phi(-t)[1 - b(t\Delta)]\mathrm{d}t\ ; \tag{2.50}$$

here, $\phi(t)$ is the Fourier transform of $f(E)$

$$\phi(t) = \int f(E)\mathrm{e}^{-2\pi iEt}\mathrm{d}E. \tag{2.51}$$

If $f(E)$ has a width $\sim \delta E$, $\phi(t)$ is appreciable over an interval $t \sim 1/\delta E$; if the number of levels inside $\delta E$, i.e. $\delta E/\Delta$ , is very large, we can approximate $b(k)$ in (2.50) for small values of the argument ($\approx 1 - 2|k|/\beta$); provided $|\phi(t)|^2$ decreases in the tails at least as $1/t^{2+\epsilon}$, we have

$$\mathrm{var}F = \frac{2}{\beta}\int \phi(t)\phi(-t)|t|\mathrm{d}t. \tag{2.52}$$

This result, first obtained by Dyson and Mehta [30], shows that the variance is independent of microscopic parameters and has a universal $1/\beta$ dependence. For instance, if $f(E) = \theta[(1 - (2E/L)^2)^{1/2}]$, where $\theta$ is the step function and $L$ the length of an energy interval, one finds $\langle F\rangle = \pi L/4\Delta = \pi\langle n\rangle/4$ and $\mathrm{var}F = 1/2\beta$, independent of $\langle n\rangle$; the implication is that this linear statistic allows the measurement of the average number of levels $\langle n\rangle$ in the interval $L$ with a mean square error of order unity.



6. If the ensemble is "unfolded" to one of uniform density, one can show that *ergodicity* is fulfilled: i.e., ensemble averages for a fixed energy are equal to energy averages for a fixed sample, up to a set of zero measure [8]. This is an important property, when quantities evaluated along one experimental spectrum are to be compared with theoretical results evaluated as ensemble averages. It is worth being a little more explicit about this point.

Suppose we have achieved stationarity by an appropriate unfolding. Consider a quantity $f(E,\xi)$, defined for the member $H^\xi$ of the ensemble and dependent on the details of (i.e. a functional of) the spectrum for that member. Let us construct the spectral average (denoted by a bar) over the interval $\delta E$ at $E$:

$$\overline{f(\xi)} = \frac{1}{\delta E}\int_{E-\frac{1}{2}\delta E}^{E+\frac{1}{2}\delta E} f(E',\xi)\,\mathrm{d}E'. \tag{2.53}$$

This is a random number, in the sense that it depends on the member $\xi$ of the ensemble. When its variance (the brackets $\langle\cdots\rangle$ denoting, as usual, an ensemble average)

$$\mathrm{var}\,\overline{f(\xi)} = \left\langle\left[\overline{f(\xi)}\right]^2\right\rangle - \left[\left\langle\overline{f(\xi)}\right\rangle\right]^2 \tag{2.54}$$

tends to zero in the limit $\delta E \to \infty$, we say that $f$ is an *ergodic* quantity [33]. Thus, for almost all members $\xi$ of the ensemble, except for a set of zero measure, (2.53) coincides with its ensemble average; i.e.

$$\overline{f(\xi)} = \left\langle\overline{f(\xi)}\right\rangle = \frac{1}{\delta E}\int_{E-\frac{1}{2}\delta E}^{E+\frac{1}{2}\delta E}\langle f(E',\xi)\rangle\,dE' = \langle f\rangle, \tag{2.55}$$

where, in the last step, we used *stationarity*. We thus have equality of spectral and ensemble averages for almost all members of the ensemble. Slutski's ergodic theorem ([33], p.20) states that a stationary random function is ergodic if and only if its two-point correlation coefficient vanishes as the separation between the two points (energies in our case) tends to infinity. This theorem has been used to study a number of ergodic properties of the GE's [8]. As an example take, for $f$, the microscopic density $\widehat{\rho}(E)$ of Eq. (2.33). Then (2.53) gives $n/\delta E$, $n$ being the number of levels inside $\delta E$, and its variance is

$$\mathrm{var}\frac{n}{\delta E} = \frac{\Sigma^2}{r^2}\rho^2. \tag{2.56}$$

Here, $r = \langle n\rangle$, $\rho$ is the mean level density, as usual, and $\Sigma^2$ the number variance, given by (2.37) for the GUE and by $\Sigma^2 \sim r$ for a Poisson process; in both cases, (2.56)$\to$0 as $r \to \infty$.

### 2.5. *Matrix ensembles with a specified density*

As was mentioned right after eq.(2.26), in numerical simulations as well as in experiments one has found a clear-cut separation between the properties of fluctuations and the secular behaviour of the density; while the former have the universal character described above, the latter is, generally, system dependent [8,22,25]. It is then appropriate to look for an ensemble that belongs to one of the above symmetry classes and has minimum information (or maximum entropy) among all those that have a prescribed density [22]. One finds, for the joint probability density of eigenvalues

$$w(E_1,\ldots,E_N) = CJ_\beta(\mathbf{E})\prod_k \mathrm{e}^{-\beta V(E_k)} = \mathrm{e}^{-\beta H}\,, \tag{2.57}$$

where the Jacobian $J_\beta$ is given by

$$J_\beta(\mathbf{E}) = \prod_{i<j} |E_i - E_j|^\beta. \tag{2.58}$$

The right-hand side of (2.57) has the form of the Boltzmann weight, in classical statistical mechanics, associated with the "Hamiltonian"

$$H = -\sum_{i<j} \ln|E_i - E_j| + \sum_i V(E_i). \tag{2.59}$$

The first term of $H$ can be interpreted as the repulsive potential between pairs of charges in a 2D electrostatic problem, while the second term is a "confining potential" which, in the Gaussian Ensembles (2.32), is a harmonic oscillator. In the large-$N$ limit, $V$ and the density $\rho$ are related through the mean-field equation [35]

$$V(E) = \int \rho(E')\ln|E - E'|\,\mathrm{d}E' + C. \tag{2.60}$$

Since $\rho(E)$ is the average of the microscopic density $\widehat{\rho}(E)$ of Eq. (2.33) , i.e.

$$\rho(E) = \frac{\int \widehat{\rho}(E)\mathrm{e}^{-\beta H}\mathrm{d}E_1\cdots\mathrm{d}E_N}{\int \mathrm{e}^{-\beta H}\mathrm{d}E_1\cdots\mathrm{d}E_N}\,, \tag{2.61}$$

the two-point correlation funtion $\rho_2$ of Eq. (2.39) is written, in ref. [34], as the functional derivative

$$\rho_2(E,E') = -\frac{1}{\beta}\frac{\delta\rho(E)}{\delta V(E')}\,. \tag{2.62}$$

In the mean-field approximation (2.60), ref. [34] thus finds that the variance of a linear statistic, Eq. (2.49), is *independent of* $V$ and has the *universal* $1/\beta$ dependence.



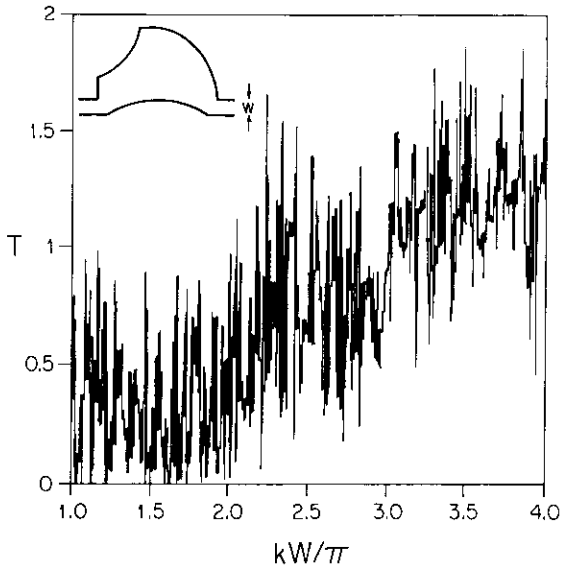


Fig. 5. Transmission coefficient of the system shown in the inset.

## 3. Random matrices in scattering problems

Various quantum mechanical scattering formalisms have been developed in the past. In order that the reader can see them in perspective, appendix A.1 presents a summary of a number of them, emphasizing those aspects that are relevant for the problems to be discussed in this section. Similarly, for the sake of completeness, appendix A.2 introduces, through simple examples, the various concepts of scattering theory that are needed in what follows.

### 3.1. *Scattering described by an equilibrated component. The circular ensembles*

The quantum-mechanical scattering produced by classically chaotic cavities has been investigated by several authors [36–46]. An example is presented in Fig. 5 [47]. The inset shows a typical system, consisting of a 2D cavity connected to the outside by a pair of leads, where confinement in the transverse direction (inside a width $W$ in each lead) produces discrete transverse modes, or *channels*. The stoppers shown inside the cavity are intended to block any direct transmission between the leads. The *scattering matrix* $S$ and the resulting transmission coefficient $T$ across the system vary as a function of the incident momentum because of the resonances occurriyng in the cavity[48]. These resonances are moderately

overlapping for one open channel ($1 < kW/\pi < 2$) and become more overlapping as more channels open up. The basic assumption (*ergodic hypothesis)* is that through these fluctuations $S$ *covers the available matrix space with uniform probability.* This concept, which reminds us of the microcanonical ensemble of classical statistical mechanics, will be developed in what follows.

A quantum scattering problem is described by the *scattering matrix* $S$ [48–51]. If the incident momentum is $k$, there are

$$N = kW/\pi \tag{3.1}$$

transmitting or running modes (channels) in each lead. The wavefunction in the leads is written as the $N$-dimensional vector

$$\Psi(x) = [\psi^1(x), \ldots, \psi^N(x)]^T, \tag{3.2}$$

the $n$-th component being a linear combination of unit-flux plane waves; i.e.

$$\psi_n(x) = a_n \frac{\mathrm{e}^{ik_n x}}{(\hbar k_n/m)^{1/2}} + b_n \frac{\mathrm{e}^{-ik_n x}}{(\hbar k_n/m)^{1/2}}, \quad n = 1, \ldots, N, \tag{3.3}$$

for the left lead, and a similar expression for the right lead, with coefficients $a'_n$, $b'_n$. In (3.3), $k_n$, the "longitudinal" momentum in channel $n$, is such that

$$k_n^2 + \left[\frac{n\pi}{W}\right]^2 = k^2. \tag{3.4}$$

The $2N$-dimensional $S$-matrix relates the incoming to the outgoing amplitudes as

$$\begin{bmatrix} b \\ a' \end{bmatrix} = S \begin{bmatrix} a \\ b' \end{bmatrix}, \tag{3.5}$$

where $a, b, a', b'$ are $N$-dimensional vectors. The matrix $S$ has the structure

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}, \tag{3.6}$$

where $r, t$ are the $N \times N$ reflection and transmission matrices for particles from the left and $r', t'$ for those from the right. The transmission coefficient referred to above is given by

$$T = \mathrm{tr}(tt^{\dagger}) \tag{3.7}$$

and the conductance of the cavity is, for "spinless" particles [52] ,

$$G = \frac{e^2}{h} T. \tag{3.8}$$



The requirement of *flux conservation(FC)* implies *unitarity* of the $S$ matrix [48–51]; i.e.

$$SS^{\dagger} = 1. \tag{3.9}$$

This is the only requirement in the absence of other symmetries. In the language of the previous chapter, this is the unitary case ($\beta = 2$). In the presence of *time-reversal invariance*, $S$, besides being unitary, is a *symmetric* matrix [48–51,53]

$$S = S^T \tag{3.10}$$

in the orthogonal case ($\beta = 1$); in the symplectic case ($\beta = 4$), for a particle with $s = 1/2$, $S$ is a $4N \times 4N$ matrix that satisfies

$$S = \hat{\Sigma} S^T \hat{\Sigma}^T, \tag{3.11}$$

where

$$\hat{\Sigma} = \begin{bmatrix} \Sigma & 0 \\ 0 & \Sigma \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0_N & -1_N \\ 1_N & 0_N \end{bmatrix}. \tag{3.12}$$

In the language of quaternions, $S$ is called a self-dual matrix.

The precise realization of the notion of "equal a priori distribution" of the $S$ matrix is the *invariant measure* associated with each symmetry class [24,53]. By definition, that measure remains invariant under an automorphism of a given class of matrices unto itself; i.e.

$$\mathrm{d}\mu_\beta(S) = \mathrm{d}\mu_\beta(S'). \tag{3.13}$$

For $\beta = 1$ we have

$$S' = U_0 S U_0^T, \tag{3.14}$$

$U_0$ being an arbitrary, but fixed, unitary matrix. Clearly, Eq. (3.14) is an automorphism of the set of unitary symmetric matrices unto itself. For $\beta = 2$,

$$S' = U_0 S V_0, \tag{3.15}$$

$U_0$ and $V_0$ being now arbitrary fixed unitary matrices. For $\beta = 4$,

$$S' = U_0 S \tilde{U}_0, \tag{3.16}$$

where $\tilde{U}_0 = \hat{\Sigma}^T U_0^T \hat{\Sigma}$. For $\beta = 2$, the resulting measure is the well known Haar's measure of the unitary group and its uniqueness is well known [28,54]. Uniqueness for the other two classes was shown in ref. [53]. Eq. (3.13) defines the *Circular (Orthogonal, Unitary, Symplectic) Ensembles (COE, CUE, CSE)*, for $\beta = 1, 2, 4$, respectively.

The eigenvalues of a unitary matrix are complex numbers $e^{i\theta_j}$ of unit modulus. The joint probability density of the *eigenphases* $\theta_j$ for the circular ensembles (CE's) was found in ref. [53] as

$$w_\beta(\theta_1,\ldots,\theta_N) = C_\beta \prod_{j<k} |e^{i\theta_j} \quad e^{i\theta_k}|^\beta; \tag{3.17}$$

its study was motivated by the convenience of having a model for the energy levels (that were associated with the eigenphases) with uniform density across the whole spectrum, rather than by its applications to scattering problems. The fluctuation properties of the eigenphases were found, for large $N$, to coincide with those of the energy levels in the Gaussian ensembles described in the last section, in the region where the density of the semicircle is a maximum [9,29].

We go back to those problems where the unitary matrix $S$ occurring in the CE's is identified with the scattering matrix. In this subsection, the possibility of "direct" processes – caused by short trajectories and giving rise to a nonvanishing average $S$ matrix [42,55–57] – is ruled out; they will be considered in the next subsection. We emphasize below the cases $\beta = 1, 2$; various generalizations to the $\beta = 4$ case can be found in ref. [45].

The quantum scattering produced by classically chaotic cavities (and subject to the restriction of the last paragraph) is found to obey the statistical properties described by the CE's. The eigenphases have been studied in [36–44]; the statistics of the $S$-matrix elements themselves was examined in refs. [37,40] and was successfully compared with the results of ref. [58]; the transmission coefficient $T$ of Eq. (3.7) was analyzed in [45,46].

Averages of products of $S$ and $S^*$ matrix elements for the CE's can be calculated without using an explicit form for $d\mu_\beta(S)$, but solely its invariant properties [59,60]. Using the notation

$$\langle f\rangle_0^{(\beta)} = \int f \mathrm{d}\mu_\beta(S), \tag{3.18}$$

with

$$\int \mathrm{d}\mu_\beta(S) = 1, \tag{3.19}$$

one finds, for example (for $\beta = 1, 2$)

$$\langle |t_{ab}|^2\rangle_0^{(\beta)} = \frac{1}{2N+2-\beta}, \tag{3.20}$$

$$\langle |t_{ab}|^2 |t_{cd}|^2\rangle_0^{(\beta)} = \frac{2(N+2-\beta)(1+\delta_{ac}\delta_{bd}) - \delta_{ac} - \delta_{bd}}{2N(2N+1)(2N+7-4\beta)}. \tag{3.21}$$



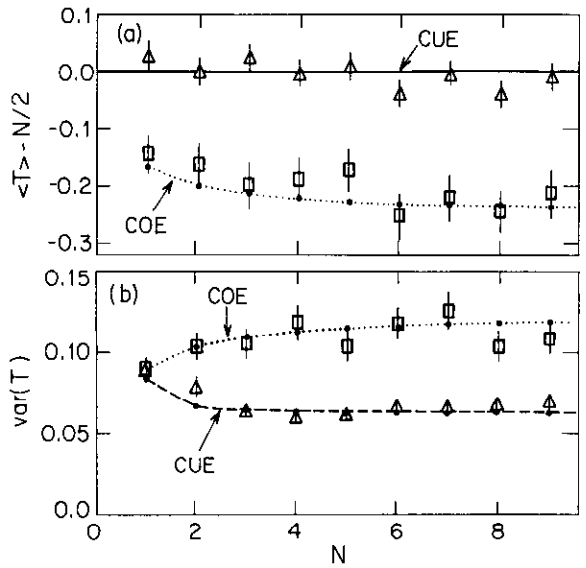


Fig. 6. Weak-localization correction and conductance fluctuations as a function of $N$ for the system shown in Fig. 5, compared with the theoretical predictions (COE, CUE) discussed in the text.

We can obtain the first and second moments of the transmission coefficient $T$ of Eq. (3.7) by performing, in the above equations, the sum over channels, with the results

$$\langle T\rangle_0^{(\beta)} - N/2 = -\delta_{1\beta}\frac{N}{4N+2} \to -\frac{1}{4}\delta_{1\beta} \tag{3.22}$$

$$\mathrm{var}(T) = \frac{N(N+1)^2}{(2N+1)^2(2N+3)} \to \frac{1}{8}, \quad \mathrm{COE} \tag{3.23}$$

$$= \frac{N^2}{4(4N^2-1)} \to \frac{1}{16}, \quad \mathrm{CUE} \tag{3.24}$$

where the limit is as $N \to \infty$. In this limit, the weak-localization correction (3.22) and the magnitude of the conductance fluctuations, var$(T)$, become universal; the latter is twice as large in the presence of time-reversal symmetry ($\beta = 1$) as in the absence of such symmetry. Fig. 6 [46] compares the above predictions for $\langle T\rangle - N/2$ and var$(T)$ with the results of a numerical simulation that uses cavities like that of Fig. 5. One can construct a random-matrix theory for the billiard Hamiltonian in terms of one of the Gaussian Ensembles discussed in the previous section and then couple the billiard to the leads in a statistical way; if the

parameters are so adjusted that $\langle S\rangle = 0$, the resulting ensemble for the $S$-matrix coincides with one of the Circular Ensembles described above (see [61–63] and also The Heidelberg approach, in appendix A.1.2).

The invariant measure for the CE's was expressed in eq. (3.17) in terms of the eigenphases; it is also convenient to express it in terms of the eigenvalues $\tau_a$ of the Hermitean matrix $tt^\dagger$, whose trace is the total transmission $T$, Eq. (3.7). The $S$-matrix of Eq. (3.6) can be expressed in the "polar representation" [64–66]

$$S = \begin{bmatrix} v^{(1)} & 0 \\ 0 & v^{(2)} \end{bmatrix} \begin{bmatrix} -\sqrt{1-\tau} & \sqrt{\tau} \\ \sqrt{\tau} & \sqrt{1-\tau} \end{bmatrix} \begin{bmatrix} v^{(3)} & 0 \\ 0 & v^{(4)} \end{bmatrix}, \tag{3.25}$$

where $\tau$ stands for the $N \times N$ diagonal matrix of the eigenvalues $\tau_a$ and the $v^{(i)}$ are arbitrary unitary matrices for $\beta = 2$, with $v^{(3)} = (v^{(1)})^T$ and $v^{(4)} = (v^{(2)})^T$ for $\beta = 1$. In the present case, the differential arc length of Eq. (2.21) is

$$\mathrm{d}s^2 = \mathrm{Tr}[dS^\dagger dS]. \tag{3.26}$$

Substituting for $S$ the form (3.25) and applying Eq. (2.22) one finds ($\beta = 1, 2$) the invariant measure [45,46]

$$\mathrm{d}\mu_\beta(S) = p_\beta(\{\tau\}) \prod_a \mathrm{d}\tau_a \prod_i \mathrm{d}\mu(v^{(i)}), \tag{3.27}$$

where the joint probability density of the $\{\tau\}$ is

$$p_\beta(\{\tau\}) = C_\beta \prod_{a<b} |\tau_a - \tau_b|^\beta \prod_c \tau_c^{(\beta-2)/2} \tag{3.28}$$

and $\mathrm{d}\mu(v^{(i)})$ denotes the invariant, or Haar's, measure on the unitary group [54] and $C_\beta$ is a normalization constant. The total transmission coefficient $T$, given by

$$T = \sum_a \tau_a, \tag{3.29}$$

is a linear statistic, in the terminology used in section 2.4.5. In the language of section 2.5, the last product in Eq. (3.28) defines the "single-particle confining potential". As remarked right after Eq. (2.62), ref. [34] shows that the limiting results (3.23), (3.24) for var$T$ as $N \to \infty$, i.e. $1/(8\beta)$, are independent of the specific form of the confining potential.

The distribution of $T$ can be obtained by integration of (3.28). As an example, Fig.7 [46] compares the resulting distribution for $N = 2$ with the same numerical simulations that were mentioned above.



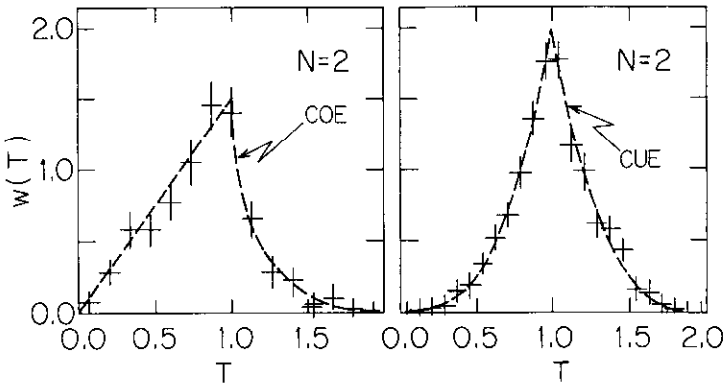


Fig. 7. Distribution of $T$ for $N = 2$; the same numerical simulation of the previous figures is contrasted with the theoretical prediction arising from COE and CUE.

### *3.2. Scattering described by a prompt and an equilibrated component*

In the last subsection we considered those scattering processes that can be described in terms of an equilibrated component only: the possibility of direct, or prompt components, giving rise to a nonvanishing averaged $S$-matrix[55], was specifically ruled out. That possibility is considered in what follows. We first illustrate the idea in a one-channel situation.

#### *3.2.1. The one-lead, one-channel case*

Consider a cavity connected to the outside by only one lead and suppose that the energy is such that there is only one open channel: scattering by the cavity just consists of reflection back to the same lead and the relevant $S$-matrix is one-dimensional.

Unitarity implies that $S$ can be written as $S = e^{i\theta}$, $\theta$ being twice the phase shift, so that $S$ is restricted to move *on* the unit circle (Fig. 8a). From analiticity, $S$ has poles only in the lower-half of the complex-energy plane (Fig. 8b and appendix A.2).

Suppose now that we perform an average of the $k$-th power of $S$, $S^k(E)$, with a *Lorentzian weighting function* centered at some energy $E_0$ and having a half-width $I$. From Cauchy's theorem one has

$$\left[\overline{S^k}\right]_{E_0,I} \equiv \int S^k(E) \frac{I/\pi}{(E-E_0)^2 + I^2} \mathrm{d}E = S^k(E_0 + iI), \tag{3.30}$$

and hence

$$\left[\overline{S^k}\right]_{E_0,I} = \left[\bar{S}\right]^k_{E_0,I}, \tag{3.31}$$

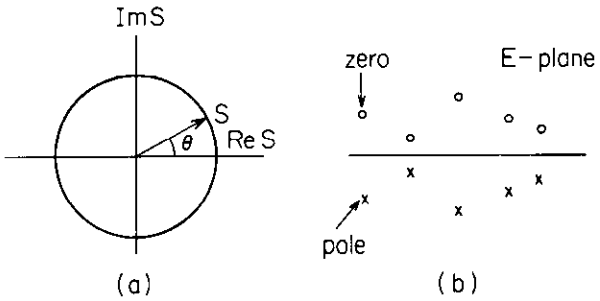


Fig. 8. Unitarity and analyticity of the $1 \times 1$ $S$-matrix for a one-lead, one-channel problem.

i.e. the average of the $k$-th power of $S$ coincides with the $k$-th power of the average of $S$. As the energy $E$ changes, $S$ of Fig. 8a moves on the unit circle and $\theta$ changes accordingly. The integral in (3.30) is over $E$; let us inquire whether it can be converted into an integral over $\theta$, in such a way that

$$\left[\overline{S^k}\right]_{E_0,I} = \int S^k \mathrm{d}P(S) = \int_0^{2\pi} \mathrm{e}^{ik\theta} p(\theta) d\theta \,, \tag{3.32}$$

with

$$\mathrm{d}P(S) = p(\theta)\mathrm{d}\theta \tag{3.33}$$

a (non-negative) measure independent of $k$. Assume (3.32) to be possible and expand $p(\theta)$ in a Fourier series; i.e.

$$p(\theta) = \sum_k a_k \mathrm{e}^{ik\theta}, \tag{3.34}$$

with $a_k = a_{-k}^*$. We find, for the expansion coefficients

$$a_{-k} = (1/2\pi) \int p(\theta)\mathrm{e}^{ik\theta}\mathrm{d}\theta = \overline{S^k}/2\pi = (\bar{S})^k/2\pi. \tag{3.35}$$

We can sum the series, with the result [67]

$$p(\theta) = \frac{1}{2\pi} \frac{1 - |\bar{S}|^2}{|S - \bar{S}|^2}, \quad S = \mathrm{e}^{i\theta}. \tag{3.36}$$

We can thus answer the question posed above: $p(\theta)$ *exists* and is *uniquely* given by Eq. (3.36) in terms of *the parameter* $\bar{S}$ *only*. The result (3.36) is valid whether $S(E)$ consists of a single pole, a regular pattern or a statistical distribution of poles; notice though that $\bar{S}$ has a numerical value that depends on the centroid



$E_0$ and the half-width $I$ of the Lorentzian of Eq. (3.30). Suppose now that $S(E)$ has resonances in the full interval $(-\infty, \infty)$ and that, as the averaging interval $2I \to \infty$ , $\bar{S}$ becomes independent of $E_0$ and $I$. In many cases of interest it also occurs that, as $I \to \infty$, Lorentzian averages become indistinguishable from more physical "box averages" (i.e. with a weight equal to $1/2I$ within the energy interval $\delta E = 2I$, and zero outside): in that case $\mathrm{d}P(S) = p_{\bar{S}}(\theta)\mathrm{d}\theta$, with $p_{\bar{S}}(\theta)$ given by (3.36), can be interpreted, in a very appealing way, as the *fraction of "time" spent within* $\mathrm{d}\theta$ by $S(E)$, as $E$ runs from $-\infty$ to $+\infty$.

Consider now an ensemble of $S(E)$ matrices, that we assume *ergodic*, in the sense explained in section 2.4.6. In particular, $\bar{S}$ will be the same for all members of the ensemble, except for a set of zero measure, and can be calculated as the ensemble average $\langle S \rangle$ at a fixed energy. Similarly, the condition (3.31) arising from analyticity, together with ergodicity, implies the relation

$$\langle S^k \rangle = \langle S \rangle^k \tag{3.37}$$

between ensemble averages, often called the *analyticity-ergodicity (AE)* requirement. The ensemble measure is thus *uniquely* given by

$$\mathrm{d}P_{\langle S \rangle}(S) = p_{\langle S \rangle}(\theta)\mathrm{d}\theta, \tag{3.38}$$

with

$$p_{\langle S \rangle}(\theta) = \frac{1}{2\pi} \frac{1 - |\langle S \rangle|^2}{|S - \langle S \rangle|^2}, \tag{3.39}$$

once $\langle S \rangle$ is specified. The ensemble depends parametrically upon the single complex number $\langle S \rangle$, *any other information being irrelevant*! Slutski's ergodic theorem, mentioned in section 2.4.6, allows us to construct the ensemble from a single realization by picking $S$ matrices at energies sufficiently far apart that their correlation coefficient is negligible.

We note in passing that (3.37) implies that a function $f(S)$ that is analytic in its argument, and can thus be expanded in a power series in $S$, must fulfill the *reproducing property* [24,56]

$$f(\langle S \rangle) = \int f(S)\mathrm{d}P_{\langle S \rangle}(S). \tag{3.40}$$

Eq. (3.40) also arises in the problem of finding the 2D electrostatic potential inside a circle, knowing its value on the boundary [68]; the Green function gives rise to the so-called *Poisson's kernel*, which is precisely our measure (3.39)!

We now compare the above results with a *microscopic* calculation. Suppose we express the one-channel $S$ matrix in terms of a $K$ matrix and construct the

latter in terms of resonances (the microscopic variables; see refs. [48,50,69,70] and also appendix A.1); i.e.

$$S = \frac{1 + iK}{1 - iK}, \tag{3.41}$$

with

$$K = \sum_i \frac{\gamma_i^2}{E_i - E}. \tag{3.42}$$

The $E_i$'s are the eigenvalues and the $\gamma_i$'s are related to the eigenvectors of a bound-state Hamiltonian. One stretch of resonances, each determined by its position $E_i$ and the amplitude $\gamma_i$, determines one $S$-matrix. A collection of stretches, following given statistical laws, like the ones studied in the previous chapter, is then used to construct an ensemble of $S$-matrices. Two ensembles, for which there is numerical evidence of ergodicity, were constructed: in the first one, the $E_i$ 's were chosen at random, the spacing $x$ between successive resonances thus following Poisson's law (2.2); in the second one, $x$ was sampled from Wigner's distribution (2.1), but with no correlation among the various spacings. In both cases, the amplitudes $\gamma_i$ were considered statistically independent Gaussian variables. The probability density of the variable $\theta$ (at a fixed energy $E$) resulting from a numerical simulation [67] for the two above ensembles (with the same value of $\langle S \rangle$) is compared, in Fig. 9, with Poisson's measure (3.39). The excellent agreement verifies the above statement that the only relevant quantity is $\langle S \rangle$ and any other information about the distribution of $S$ is irrelevant. A *central-limit theorem (CLT)* of a novel kind is responsible for the insensitivity of $p(\theta)$ on details other than $\langle S \rangle$. Poisson's kernel (3.39) for $S$ implies Cauchy's distribution for the quantity $K$ appearing in (3.41). Thus, viewing $K$ as the sum over resonances (3.42), our discussion implies that, in the limit when the number of resonances grows, the resulting $K$ becomes eventually distributed according to Cauchy's distribution; this occurs no matter what is the distribution of the resonances, as long as one has ergodicity! As a verification of this general theorem, the *explicit* evaluation of the distribution of $K$ in the particular case of randomly distributed $E_i$'s and statistically independent (but with an arbitrary individual distribution) $\gamma_i$'s is performed in appendix B: the standard central-limit theorem for independent variables [21], leading to a Gaussian distribution, does not apply, because the $1/E$ structure of the individual terms building up $K$ gives divergent second and higher moments; in fact, in this particular situation of independent levels, the result is a particular case of the distributions studied by Levy [72,73].



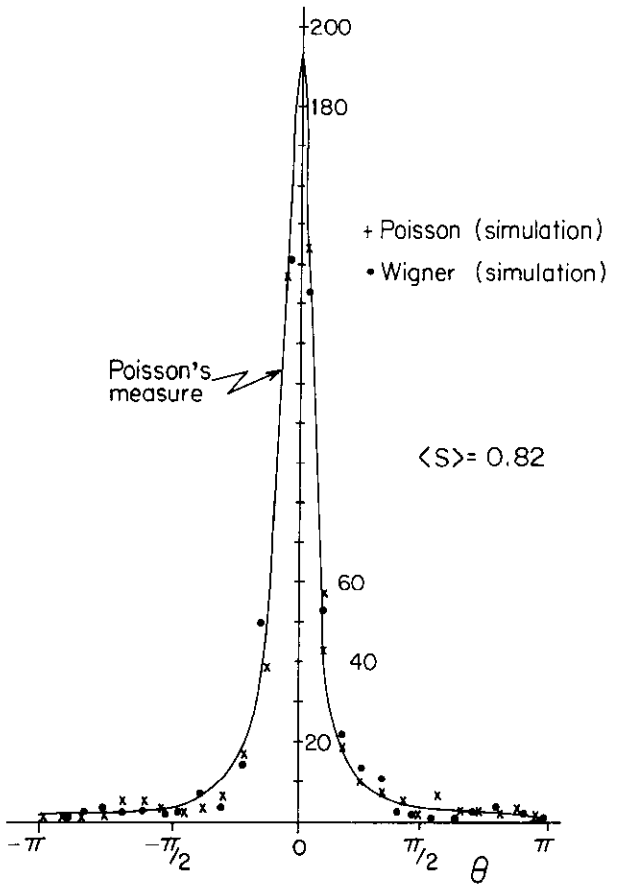


Fig. 9. Comparison of Poisson's measure with numerical simulations.

*3.2.2. The multichannel case*

We now consider $S$ matrices of dimensionality $n$, that can describe, say, a one-lead problem with $n$ channels or a two-lead problem with $N = n/2$ ($n$ even) channels each. The Argand diagram of Fig. 8a has to be generalized to include the axes $\mathrm{Re}\, S_{11}$, $\mathrm{Im}\, S_{11}$, $\mathrm{Re}\, S_{12}$, $\mathrm{Im}\, S_{12}$, ..., $\mathrm{Re}\, S_{nn}$, $\mathrm{Im}\, S_{nn}$; $S$ is restricted to move on the surface determined by unitarity ($SS^\dagger = I$) and, for $\beta = 1$, symmetry ($S = S^T$).

The AE requirement (3.37) is now generalized to

$$\langle (S_{a_1 b_1})^{n_1} \cdots (S_{a_k b_k})^{n_k} \rangle = \langle S_{a_1 b_1} \rangle^{n_1} \cdots \langle S_{a_k b_k} \rangle^{n_k}\,; \tag{3.43}$$

notice that this expression involves only $S$, but no $S^*$ matrix elements. Similarly, if $f(S)$ is a function that can be expanded as a series of nonnegative powers of $S_{11}, \ldots, S_{nn}$ (analytic in $S$), we must have the reproducing property (3.40).

Our starting point is the invariant measure $d\mu_\beta(S)$ that was introduced in the last subsection, eq. (3.13). The average of $S$ evaluated with that measure vanishes, so that the prompt, or direct, components vanish. It is easy to check that the AE requirements (3.43) or, equivalently, the reproducing property (3.40), is satisfied exactly for the invariant measure. Ensembles that contain more information than the invariant one are constructed by multiplying the latter by appropriate functions of $S$. Just as in the previous section, we relate the probability density $p_{\bar{S}}^{(\beta)}(S)$ to the differential probability through (notice that we require $\langle S \rangle = \bar{S}$)

$$\mathrm{d}P_{\bar{S}}^{(\beta)}(S) = p_{\bar{S}}^{(\beta)}(S)\mathrm{d}\mu_\beta(S) \tag{3.44}$$

and require the fulfillment of the AE conditions. Mathematicians know a solution to this problem, which, for arbitrary dimensionality of the $S$-matrix, is again known as *Poisson's kernel* and is given by [24,56]

$$p_{\bar{S}}^{(\beta)}(S) = V_\beta^{-1} \frac{[\det(I - \bar{S}\bar{S}^\dagger)]^{(\beta n+2-\beta)/2}}{|\det(I - S\bar{S}^\dagger)|^{\beta n+2-\beta}}, \tag{3.45}$$

Here, $V_\beta$ is a normalization factor. We notice that, for $n = 1$, eq. (3.45) reduces to (3.36).

In the case $n > 1$ there is, though, a very important difference with respect to the $n = 1$ case, in that AE and reality of the solution do not fix $p_{\bar{S}}(S)$ uniquely. Indeed, in general (the $1 \times 1$ case being exceptional), we expect the matrix $\bar{S}$ to be insufficient to characterize the full distribution when, in addition to the prompt and equilibrated components, there are other contributions associated with different time scales [69]. Out of all possibilities, though, Poisson's kernel occupies a very special place: its entropy

$$\mathcal{S}[p] \equiv -\int p_{\bar{S}}(S) \ln p_{\bar{S}}(S) \mathrm{d}\mu(S) \tag{3.46}$$

is larger than or equal to that of any other kernel satisfying AE and for the same $\bar{S}$. We can say that Poisson's kernel describes those physical situations in which, having imposed the general requirements of flux conservation, time-reversal invariance (when applicable) and AE, most of the details are irrelevant, except for the average matrix $\bar{S}$.

Before finishing our discussion we mention, in relation with the $S$-matrix model (3.45), that it has been found advantageous to map the physical $S$-matrix unto another $S$-matrix, $S'$, for which the probability density reduces to a constant and thus the differential probability for $S'$ is given by the invariant measure; i.e.

$$\mathrm{d}P(S') = \mathrm{d}\mu(S'). \tag{3.47}$$

The transformation in question is, for $\beta = 1$ ([57];[24], p.84)

$$S' = R(S - \bar{S})(I - \bar{S}^* S)^{-1}(R^*)^{-1}, \tag{3.48}$$

where the matrix $R$ satisfies the relation

$$R(I - \bar{S}\bar{S}^\dagger)R^\dagger = I. \tag{3.49}$$

Therefore, if $\bar{S} = 0$, $S$ is described by one of the CE's of the previous subsection; if $\bar{S} \neq 0$, it is $S'$ that is described by a CE: in particular, the eigenphases of $S'$ satisfy the distribution given by Eq. (3.17), its transmission coefficient $T'$ has the average and variance of (3.22) and (3.23) and its polar parameters $\tau_a'$ are distributed according to Eq. (3.28).

Ref. [42] analyzes a scattering problem described in terms of a $2 \times 2$ $S$ matrix and a nonzero $\bar{S}$ in terms of Poisson's distribution (3.45) and finds a satisfactory agreement. Further studies in this direction are now in progress [75].

Refs. [69–71] deal with the problem of the present subsection by constructing the $S$-matrix in terms of a Hamiltonian, for which a GOE is assumed; this was described right after Eq. (3.24) and in appendix A.1.2. The two-point function found in [69], together with Slutski's theorem [33], is used in ref. [76] to study the ergodic properties of the $S$-matrix ensembles. It is significant that both the ensemble of refs. [69–71] and Poisson's measure discussed above are of maximum entropy, but at different levels: one at the level of the Hamiltonian and the other at the level of the $S$-matrix. The relation between the two ensembles has recently been studied in ref. [74], which then explains the coincidences noticed in the past [56,57] for one- point functions between the two approaches.

### *3.3. Scattering in disordered media. Diffusion processes and localization*

In the confined geometry shown in Fig. 10 the portion of length $L$ is a disordered medium that scatters waves incident upon it. The diffusion time across the system is the relevant characteristic time. We could imagine an experiment performed with microwaves or, if the disordered system is a mesoscopic device, a situation where the interest is in the electronic transport. The goal is to study the *coherent multiple scattering* produced by the disordered part and its statistical properties across a collection (ensemble) of samples that differ, from one another, in their microscopic configuration.

Quantities of interest are, e.g., the average and fluctuations of the transmission coefficient $T_{ab} = |t_{ab}|^2$ ($t$ being given in Eq. (3.6)), the correlation between

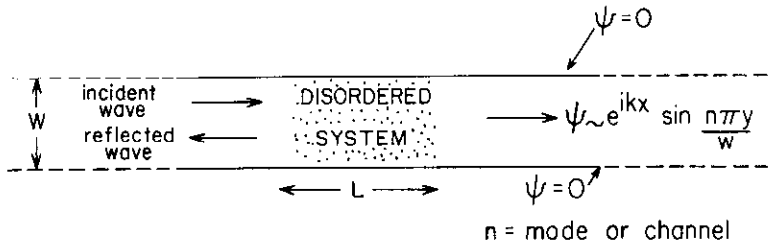


Fig. 10. Disordered system studied in the text.

pairs of such coefficientes, the average and fluctuations of the total transmission coefficient $T$ defined in Eq. (3.7), etc.

For definiteness we use, in what follows, the terminology associated with the electronic transport problem. Ideally, we consider the system at zero temperature, so that we have no inelastic processes and can concentrate on the coherent properties of the scattering process.

*3.3.1. The isotropic model: a global approach*

Consider the polar representation (3.25) of the $S$-matrix associated with the *full sample*. In the isotropic model [66,77,78] the unitary matrices $v^{(j)}$ are distributed according to the *invariant (or Haar's) measure* $d\mu(v^{(j)})$ *of the unitary group*: this corresponds to the intuitive notion of "equal a priori probability", and thus maximum entropy, or minimum information, for the "angular" part of the *global* $S$-matrix. Moreover, the matrices $v^{(j)}$ are chosen to be statistically independent of one another and of the $\tau$'s. The specific choice (3.28) for the distribution of the $\tau_a$'s would take us back to the CE's of section 3.1; no particular choice is made, though, for the time being; we come back to this problem later on.

It is clear that the average $\langle S\rangle$ vanishes in the isotropic model, meaning that there are no prompt components, in the language introduced earlier. Although this may be a good approximation for $\langle t\rangle$, it is an oversimplification for $\langle r\rangle$, for which the contribution of short paths may be important [44]. We see below some of the consequences of this assumption.

We first write the reflection coefficient $R_{ab}$ as

$$R_{ab} = |r_{ab}|^2 , \tag{3.50}$$

where $r$ is given by Eqs. (3.6) and (3.25) as

$$r = -v^{(1)}\sqrt{1-\tau}\, v^{(3)}. \tag{3.51}$$

As we mentioned in section 3.1, averages of products of matrix elements of a unitary matrix can be evaluated explicitly [60] for the invariant measure. It is then a simple exercise to calculate the ratio of the reflection coefficient back to the same channel to that leading to a different channel; one finds the simple answer (see [65,79] and, for $\beta = 4$, [80])

$$\frac{\langle R_{aa}\rangle^{(\beta)}}{\langle R_{a\neq b}\rangle^{(\beta)}} = 2/\beta. \tag{3.52}$$

We thus see that in the orthogonal case ($\beta = 1$) backward scattering to the same channel is enhanced by a factor 2, as compared with scattering to any other channel. This is precisely the prediction of weak-localization theory, where the argument is that the various paths contribute with random phases, except for a path and its time-reversed one, which contribute coherently and give rise to a factor 2 in the backward direction. The same argument predicts that when time-reversal symmetry is destroyed by a magnetic field, the above-mentioned enhancement is absent; this is precisely what eq. (3.52) shows for the unitary case ($\beta = 2$).

One can also calculate the correlation coefficient of the transmission coefficients $T_{ab}$, $T_{cd}$. The structure of the result, with regards to the channel-index dependence, can again be evaluated using the techniques of invariant integration; one finds [65,79]

$$\begin{aligned}\langle T_{ab}T_{a'b'}\rangle^{(\beta)} = {} & [A_N\langle T^2\rangle^{(\beta)} - B_N\langle T_2\rangle^{(\beta)}]\delta_{aa'}\delta_{bb'} \\ & + [A_N\langle T_2\rangle^{(\beta)} - B_N\langle T^2\rangle^{(\beta)}](\delta_{aa'} + \delta_{bb'}) \\ & + [A_N\langle T^2\rangle^{(\beta)} - B_N\langle T_2\rangle^{(\beta)} - C_N(\langle T\rangle^{(\beta)})^2].\end{aligned} \tag{3.53}$$

Here we have defined

$$T_k = \sum_a \tau_a^k, \tag{3.54}$$

$$A_N = \frac{N^2+1}{N^2(N^2-1)^2}, \quad B_N = \frac{2}{N(N^2-1)^2}, \quad C_N = 1/N^4. \tag{3.55}$$

The averages appearing on the r.h.s of (3.53) are independent of the channel indices; their explicit value needs a specific model for the statistical distribution of the $\tau_a$'s.

In perturbative calculations [81] one finds, for the correlation between pairs of transmission coefficients, three types of terms, which, for quasi-1D systems, have essentially the structure provided by the $\delta$ functions of Eq. (3.53); the difference is that the Kronecker deltas are replaced by functions which peak at those wave vectors that satisfy the appropriate Kronecker deltas in our calculation, but decay over some distance in momentum space. A similar comment applies to the channel dependence of the backward scattering (3.52). The assumption of isotropy thus leads to quite important consequences; we see, however, the necessity to improve it.

What can we say about the joint distribution of the $\tau_a$'s? In the metallic regime, $L \gg l$, we need a choice that describes the diffusion that has taken place between the two ends of the sample. Within the present global approach the following scheme has been developed [77,78,82]. Consider a system with a specified geometry; assuming that the relevant physical information is contained in the density $\rho(\tau)$, it is proposed to look for the distribution that has maximum entropy among those that have that density $\rho(\tau)$. This approach is analogous to the one outlined in section 2.5. The differential probability for the $S$ matrix can be written as

$$dP(S) = p(S)d\mu(S) \tag{3.56}$$

$$= C_\beta \prod_{a<b} |\tau_a - \tau_b|^\beta \prod_c e^{-\beta V(\tau_c)} \prod_c d\tau_c \prod_i d\mu(v^{(i)}), \tag{3.57}$$

where the "confining potential" $V(\tau)$ has to be chosen so as to reproduce the required $\rho(\tau)$. Since the total transmission coefficient $T$ is given by the linear statistic (3.29), ref. [34] shows that the structure (3.57) implies $\mathrm{var}T = 1/8\beta$, independently of the confining potential (universal conductance fluctuations).

*3.3.2. The evolution of the statistical distribution: a local approach*

We now look at the "evolution", with increasing sample length $L$, of the statistical distribution, that was not available in the above analysis.

So far we have described a scattering process through its $S$-matrix. As explained in appendix A.2, one can, alternatively, employ the *transfer matrix* $M$ [51], that relates the coefficients on the two sides of the scatterer. The scattering and transfer matrices are related, so that the scattering process can be described in terms of either one. In the present context we shall find the use of the transfer matrix particularly advantageous, because of its *multiplicativity property*: the transfer matrix for a collection of scatterers (with a potential free region between adjacent ones) is the product of the individual transfer matrices. The collection of transfer matrices has the properties of a *group*, which is not the case for the scattering matrices.

Just as the $S$-matrix was defined in eq.(3.5), the $2N \times 2N$-transfer matrix $M$ is defined through the relation

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = M \begin{bmatrix} a \\ b \end{bmatrix}, \quad M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{3.58}$$

where $\alpha, \beta, \gamma, \delta$ are $N \times N$ matrices.

The requirement of *flux conservation* imposes on $M$ the restriction [64–66]

$$M \Sigma_z M^\dagger = \Sigma_z \ , \tag{3.59}$$



where

$$\Sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.60}$$

is the $2N$-dimensional generalization of the usual Pauli matrix $\sigma_z$, 1 being the $N \times N$ unit matrix. We thus see that the transfer matrices $M$ satisfying flux conservation form the pseudounitary group $U(N, N)$. In the terminology of section 2, this is the unitary case ($\beta = 2$).

In the orthogonal case ($\beta = 1$), the transfer matrix $M$ must also satisfy the requirement [64–66]

$$M^* = \Sigma_x M \Sigma_x \ , \tag{3.61}$$

where $\Sigma_x$ is the $2N$-dimensional generalization of the Pauli matrix $\sigma_x$. Relations (3.59) and (3.61) are illustrated in appendix C.2.2 in a 1D ($N = 1$) case. One can check that these relations define the real symplectic group $Sp(2N, R)$. Eq. (3.61) implies, for the $M$ of (3.58), the structure

$$M = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}. \tag{3.62}$$

A transfer matrix can be written in the *polar representation* [64–66]

$$M = \begin{pmatrix} v^{(2)} & 0 \\ 0 & [v^{(4)}]^\dagger \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\tau}} & \sqrt{\frac{1-\tau}{\tau}} \\ \sqrt{\frac{1-\tau}{\tau}} & \frac{1}{\sqrt{\tau}} \end{pmatrix} \begin{pmatrix} v^{(3)} & 0 \\ 0 & [v^{(1)}]^\dagger \end{pmatrix}, \tag{3.63}$$

in terms of the same parameters used in Eq. (3.25) for the $S$-matrix. The "radial parameters" $\lambda_a$ defined through

$$\tau_a = \frac{1}{1 + \lambda_a} \tag{3.64}$$

are sometimes useful.

The multiplicativity property of the transfer matrix allows a *local* approach to the problem: the full system is built from thin slices, or *building blocks*, of length $\delta L$, that we assume small on a macroscopic scale, but still thick enough to contain many impurities. Consider then a system of length $L$, described by the transfer matrix $M_L$. We add to it a thin slice (with the characteristics indicated above), described by the transfer matrix $M_{\delta L}$. Because of the multiplicativity property mentioned above, the resulting transfer matrix $M_{L+\delta L}$ can be written as

$$M_{L+\delta L} = M_{\delta L} M_L. \tag{3.65}$$

Assuming the two pieces to be statistically independent, we can write the resulting probability density in terms of the individual ones as the *convolution* [66]

$$p_{L+\delta L}(M) = \int p_L[M_{\delta L}^{-1}M]p_{\delta L}(M_{\delta L})\mathrm{d}\mu(M_{\delta L}), \tag{3.66}$$

where $d\mu(M)$ is the invariant measure associated with the group of transfer matrices.

Notice the similarity between the "combination requirement" (3.66) and the Smoluchowsky equation for Markovian processes: for instance, for a Brownian particle, that equation gives the velocity distribution at time $t+\delta t$ in terms of the distribution at time $t$ and the transition probability. One can then convert such an integral equation into a differential equation of the Fokker–Planck type [83,84]. A similar procedure can be followed in the present context.

In refs. [65,66,80] the statistical distribution for the building block is chosen on the basis of a maximum-entropy criterion, the physical restriction being *a given value of the elastic mean-free-path* $l$. The resulting $p_{\delta L}(M_{\delta L})$ is *isotropic*, just as in the models described in the first part of this subsection. It can be proved that the property of isotropy is preserved under convolution, so that the final distribution for the full sample is again isotropic. The ansatz for the building block that we just described is introduced in Eq. (3.66), which is then expanded in powers of $\delta L$; in the limit $\delta L \to 0$ one finds, for the joint probability density $w^{(\beta)}(\lambda)$ of the $\lambda_a$'s of Eq. (3.64), the Fokker–Planck or diffusion equation

$$\frac{\partial w_s^{(\beta)}(\lambda)}{\partial s} = \frac{2}{\beta N + 2 - \beta}\sum_{a=1}^{N}\frac{\partial}{\partial\lambda_a}\left[\lambda_a(1+\lambda_a)J_\beta(\lambda)\frac{\partial}{\partial\lambda_a}\frac{w_s^{(\beta)}(\lambda)}{J_\beta(\lambda)}\right], \tag{3.67}$$

where

$$s = L/l \tag{3.68}$$

is the length of the sample measured in units of the mean-free-path and $J_\beta(\lambda)$ is the Jacobian of Eq. (2.58), with $\lambda$ replacing $E$. The initial condition associated with (3.67) is the one-sided delta function

$$w_{s=0}^{(\beta)}(\lambda) = \delta(\lambda). \tag{3.69}$$

It is interesting to notice that one can find a *generalized central-limit theorem* that gives rise to the same result (3.67), (3.69), thus providing a nice interpretation of the information-theoretic argument presented above [85].

For a large number of channels, $N \gg 1$, one can expand the expectation value $\langle T\rangle$ of the total transmission coefficient in decreasing powers of $N$ [65,87]. For instance, for $\beta = 1$ one finds

$$\langle T\rangle_L = \frac{N}{1+L/l} - \frac{1}{3}\left(\frac{L/l}{1+L/l}\right)^3 + O(1/N). \tag{3.70}$$

In the metallic regime, $l \ll L \ll Nl$, the first term of (3.70) gives a *diffusive* or *ohmic* behaviour; the second term describes coherent *weak-localization*, of a quantum-mechanical origin. A similar expansion for $\mathrm{var}T$ gives [65,87,80]

$$\mathrm{var}T = \frac{2}{15\beta} + \cdots, \tag{3.71}$$

a universal result independent of $N, L, l$. This value of $\mathrm{var}T$ should be contrasted with that arising from the global approach, i.e. $\mathrm{var}T = 1/8\beta$, which in turn is the same as that for a cavity, Eqs. (3.23), (3.24).

The above results are consistent with microscopic Green function calculations [88] performed for quasi-1D systems.

The diffusion equation (3.67) for $\beta = 2$ was solved exactly in ref. [89], while ref. [90] gives a solution for arbitrary $\beta$ using a linearization procedure. In ref. [90] the second and fourth cumulants of the $T$-distribution are studied, while cumulants of arbitrary order are analyzed, for $\beta = 2$, in ref. [91].

The model for the building block is improved in ref. [86]. A modified argument is presented here. Since the transfer matrix of the building block is expected to be close to the unit matrix, we write

$$M_{\delta L} = I + \epsilon\,. \tag{3.72}$$

We assume a statistical model such that

$$\langle\epsilon_{ab}^{jk}\rangle_{\delta L} = 0, \tag{3.73}$$

$$\langle\epsilon_{ab}^{jk}\epsilon_{cd}^{lm}\rangle = \sigma_{ab,cd}^{jk,lm}\delta L + \cdots, \tag{3.74}$$

while $\langle\epsilon^3\rangle_{\delta L}$, etc, are higher order in $\delta L$. We consider specifically the $\beta = 1$ case and assume the $\epsilon$ matrix to have the four-block structure of eq. (3.62); the upper indices in eqs. (3.73), (3.74) take on the values 1, 2 and indicate the block, while the lower indices run from 1 to $N$ and denote the row and column in that block. Appendix C shows that the evolution (with increasing length) of the expectation value of a function $F(M)$ obeys the Fokker–Planck-like equation

$$\frac{\partial\langle F(M)\rangle_L}{\partial L} = \frac{1}{2}\sigma_{ab,cd}^{jk,lm}\langle M_{be}^{kn}M_{df}^{mp}\frac{\partial^2 F}{\partial M_{ae}^{jn}\partial M_{cf}^{lp}}\rangle_L\,. \tag{3.75}$$

Here, a summation over repeated indices is understood. The $\sigma$'s play the role of generalized "diffusion coefficients". A model for them is presented in appendix

C, where the resulting expression for the diffusion equation is also given. There, one reduces the full set of coefficients to $\sigma_{ab} = 1/l_{ab}$, $\sigma'_{ab} = 1/l'_{ab}$, the inverse mean-free-paths for backward and forward scattering, from channel $b$ to $a$. It would be interesting to investigate whether the solution of the diffusion equation (3.75) for the $p_{\delta L}$ associated with a thin slab is one of maximum entropy with the restrictions (3.73), (3.74), just as it happened in relation with the diffusion equation (3.67).

For the average transmission coefficients of a *thin slab* one obtains the expressions

$$\langle T_{aa}\rangle_L = 1 - [\sum_b \sigma_{ab} + \sum_{b(\neq a)} \sigma'_{ab}]L + \cdots, \tag{3.76}$$

$$\langle T_{a\neq b}\rangle_L = \sigma'_{ab}L + \cdots. \tag{3.77}$$

which are physically reasonable. The evolution of the average transmission amplitude $t_{ab}$ is given by

$$\partial_L \langle t_{ab}\rangle_L = -\sigma_a \langle t_{ab}\rangle_L, \quad \langle t_{ab}\rangle_{L=0} = \delta_{ab}, \tag{3.78}$$

so that

$$\langle t_{ab}\rangle_L = \mathrm{e}^{-\sigma_a L}\delta_{ab}. \tag{3.79}$$

($\sigma_a = \sum_b \sigma_{ab}$). These results describe a *gradual* phase "randomization" as the length $L$ increases. Whether one reaches precisely isotropy for $L \gg l$ is not clear; there is evidence, though, that the phases and the $\lambda_a$'s retain an important statistical correlation [86,92]

It is interesting to mention that in the *one-parameter model*

$$\sigma_{ab} = \frac{1+\delta_{ab}}{(N+1)l}, \tag{3.80}$$

the joint probability density $w(\lambda)$ of the $\lambda_a$'s of eq. (3.64) obeys precisely the diffusion equation (3.67) (with $\beta = 1$), and all the consequences mentioned below that equation follow. It appears that for quasi-1D systems all channels are so thoroughly mixed that approximating the various mean-free-paths by a single one, $l$, is a good approximation.

The analysis of refs. [69,71] was generalized in refs. [61,62] in order to treat the present problem. The Hamiltonian associated with small portions of the system of dimensions $\sim l$ are modelled by independent GE's. Each block is then connected to the adjacent ones, thus giving rise to a diffusion process across the sample. Finally, the system is connected to the external channels to find the total $S$ matrix, from which the transmission coefficient and its statistical properties are evaluated.



The analysis is carried out, in 2D, for arbitrary width $W$ and length $L$ of the system. For one block, as well as in the quasi-1D case, the results are consistent with the ones discussed above.

## Appendix A. Scattering theory

### *Appendix A.1. A survey of quantum-mechanical scattering theories*

Various scattering theories have been developed in the past. We summarize here a few aspects of them that are relevant to the present course. A more comprehensive information can be found in the cited literature.

#### *Appendix A.1.1. Scattering formalisms*

*1. Wigner's R-matrix theory.* The motivation for the development of this formalism was the description of nuclear reactions. An excellent comprehensive presentation can be found in [50] and in the references contained therein.

Wigner approached the problem in a vein similar to the description of the scattering of electromagnetic waves that reach a cavity through a number of waveguides or *channels*. In the context of nuclear physics the term channel refers to a pair of particles in a definite quantum state [93–95] and the equivalent of the cavity is the compound system formed by the target and the projectile. It is interesting that in the *quantum dots* contemplated in modern mesoscopic physics this terminology is even closer to the experimental setup; in fact, a channel refers to a transverse mode in one of the leads, that really act as waveguides and are connected to a real cavity.

The solution of the Schrödinger equation inside the cavity with boundary conditions at its surface yields discrete eigenvalues $E_i$ and eigenfunctions $\Psi_i$. Inside the cavity, the wavefunction for the scattering problem at energy $E$ is expanded in terms of the $\Psi_i$'s and then joined smoothly with the external wavefunction to find the *scattering matrix* $S$; this is done by imposing continuity of the logarithmic derivative at the boundary, which yields the fundamental quantity $R_{cc'}$ (where $c, c'$ label channels), known as the *R-matrix* and given by

$$R_{cc'} = \sum_i \frac{\gamma_{ic}\gamma_{ic'}}{E_i - E}. \tag{A.1}$$

Here the *reduced amplitude* $\gamma_{ic}$ is proportional to the overlap of the wavefunction $\Psi_i$ with the wavefunction for channel $c$ at the boundary. Consider, for simplicity, the special case of one-channel $s$-wave scattering by a $3D$ rotationally invariant potential of range $a$, or, equivalently, scattering by a $1D$ potential of range $a$

with an infinite reflecting barrier at the origin. The relation between the $S$ and $R$ matrices is then given by

$$S = e^{-2ika}\frac{1+ikaR}{1-ikaR}, \tag{A.2}$$

where $k$ is the incident wavenumber.

The quantity sometimes called the *K-matrix* (denoted by $Q$ in [50]) is related to $S$ by

$$S = \frac{1+iK}{1-iK}. \tag{A.3}$$

*2. The Lippman–Schwinger equation of potential scattering.* The reader is referred to refs. [49,96] for an excellent presentation of this topic.

The Hamiltonian $H$ is split into the kinetic energy $H_0$ and the interaction $V$. The eigenfunction $\psi_{\mathbf{k}}$ of $H$ is related to the eigenfunction $\phi_{\mathbf{k}}$ of $H_0$ (a plane wave) through the Lippman–Schwinger integral equation

$$|\psi_a^{(\pm)}(E)\rangle = |\phi_a(E)\rangle + \frac{1}{E^{(\pm)}-H_0}V|\psi_a^{(\pm)}(E)\rangle. \tag{A.4}$$

Here $E$ is the total energy, $a$ (which will play the role a channel index) the direction of $\mathbf{k}$ and the wavefunctions are normalized in the sense of a Dirac delta-function in these two variables; $E^{(\pm)} = E \pm i\epsilon$ (with $\epsilon \to 0$) insures outgoing(+) or incoming(-)-wave boundary conditions. Iteration of Eq. (A.4) generates Born series of potential scattering.

The *on-shell S-matrix*, defined through the relation

$$\langle\psi_a^{(-)}(E)|\psi_b^{(+)}(E')\rangle = S_{ab}(E)\delta(E-E'), \tag{A.5}$$

is a *unitary* matrix. The *T-matrix*, defined by

$$S_{ab}(E) = \delta_{ab} - 2\pi i T_{ab}(E), \tag{A.6}$$

is in turn given by

$$T(E) = V + V\frac{1}{E^{(+)}-H_0}T. \tag{A.7}$$

*3. Feshbach's projector-operator formalism.* Feshbach divides Hilbert's space into two orthogonal parts by means of the projector operators $P$ and $Q$ (with $P+Q=1, PQ=QP=0$), with the idea that $P$ projects unto the *prompt* (in time) component of the wave function, that has a smooth energy dependence, while $Q$ projects unto the time-delayed component, with a more complicated energy dependence [97,98]. In various nuclear physics applications $Q$ contains the bound-state-in-the-continuum (BSC)[69] that occur because of the many-body nature of the system and $P$ contains the rest. Choosing $P$ to project outside and $Q$ inside a cavity with a given geometry (a natural choice for a quantum dot), one can reproduce Wigner's $R$-matrix theory.



The Schrödinger equation can be written as the two coupled equations

$$(E-H_{PP})P|\Psi\rangle = H_{PQ}Q|\Psi\rangle \tag{A.8}$$

$$(E-H_{QQ})Q|\Psi\rangle = H_{QP}P|\Psi\rangle, \tag{A.9}$$

where the notation $H_{AB} = AHB$ was used. Using the Lippman–Schwinger equation described above (actually its generalization, known as the "two-potential formula"), one can find the $T$-matrix as

$$T = T^{(pot)} + H_{PQ}\frac{1}{E^{(+)}-H_{QP}\frac{1}{E^{(+)}-H_{PP}}H_{PQ}}H_{QP}. \tag{A.10}$$

*Appendix A.1.2. Statistical scattering*

The development of statistical scattering theories was motivated in the past by nuclear scattering phenomena that exhibit statistical properties. Many of those ideas now find an application to the description of mesoscopic devices. We mention here some of these approaches.

*1. The M.I.T. approach.* A representative paper on this approach is ref. [99]; more references can be found in [8]. Using Feshbach's formalism, the $S_{cc'}$ matrix element is written in the "optical-background representation"

$$S_{cc'} = \langle S_{cc'}(E)\rangle - i\sum_q \frac{g_{qc}(E)g_{qc'}(E)}{E-\mathcal{E}_q}, \tag{A.11}$$

that contains explicitly a prompt and a time-delayed component. The various residues and poles occurring in the above expression are assumed to be statistically independent in the large $\Gamma/\Delta$ limit, attained when the number of open channels is very large. Unitarity is not guaranteed in this approach, except on average. Average cross sections can be expressed entirely in terms of the optical matrix $\langle S\rangle$.

*2. The Heidelberg approach.* As is mentioned in the last paragraph of section 3.2, refs. [69,71] are representative of this approach; more references can be found therein. In this approach the $S$-matrix is constructed in terms of bound states (for which a GOE is assumed) that are coupled to the various channels by means of coupling amplitudes $\gamma_{ic}$, for which a Gaussian distribution is assumed. Unitarity is guaranteed in this approach. In the case in which only a prompt and an equilibrated contribution are considered, the full statistical distribution of the $S$

matrix and all quantities derived from it can be parametrized in terms of the optical matrix $\langle S\rangle$. An important feature of this approach is that the $S$-matrix distribution is covariant [61–63], in the sense that, under the transformation (3.14) (for $\beta = 1$) it fulfills the relation

$$dP_{\langle S\rangle}(S) = dP_{U_0\langle S\rangle U_0^T}(U_0 S U_0^T). \tag{A.12}$$

For $\langle S\rangle = 0$ we have

$$dP_0(S) = dP_0(U_0 S U_0^T), \tag{A.13}$$

so that covariance means invariance; from the uniqueness theorem of the invariant measure, (A.13) must then coincide with the COE.

*3. Direct study of the statistical distribution of the S matrix.* This approach studies those situations in which one has a prompt and an equilibrated component only, so that the full statistical problem can be parametrized in terms of an optical matrix $\langle S\rangle$, which then plays the role of the *relevant parameter*, in the terminology of section 1. The distribution of the $S$ matrix is constructed directly, i.e., without going through a Hamiltonian, in order to reflect this fact: it is required to carry minimum information or maximum entropy, once $\langle S\rangle$ is specified [56,59,100] and a number of important physical requirements, like ergodicity of the ensemble and analyticity of $S$, are fulfilled. The theory is guaranteed to be unitary and fulfills the covariance property of the previous paragraph. It is described in more detail in section 3.2.

*Appendix A.2. Simple examples of quantum-mechanical scattering problems*

For the sake of completeness we illustrate here, by means of simple examples, the notions of scattering theory that are needed throughout the course.

*Appendix A.2.1. A $1\times1$ scattering matrix: illustration of the concepts of unitarity and analytic structure in the complex energy plane*

In the present 1D example, shown in Fig. 11 (see ref.[48] for a detailed discussion), one has an infinite wall at $x = 0$ and a $\delta$-potential at $x = a$. The solution of the Schrödinger equation

$$\left(\frac{d^2}{dx^2} + k^2\right)\psi(x) = u_0\delta(x-a)\psi(x) \tag{A.14}$$

has the form

$$\psi(x < a) = A\sin kx \tag{A.15}$$

$$\psi(x > a) = e^{-ikx} + Se^{ikx}. \tag{A.16}$$



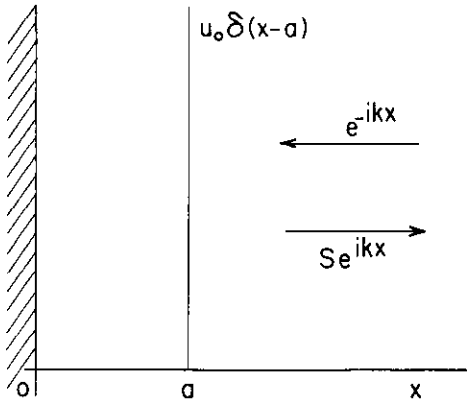


Fig. 11. The one-dimensional scattering problem studied in the text, giving rise to a $1\times1$ $S$-matrix.

The coefficient $S$ of the outgoing wave is the *scattering matrix*, a $1\times1$ matrix in this example. The wavefunction, eqs. (A.15) and (A.16), is continuous at $x = a$, while the slope has the discontinuity

$$\left[\frac{d\psi}{dx}\right]_{x=a-\epsilon}^{x=a+\epsilon} = u_0\psi(a). \tag{A.17}$$

With these two boundary conditions one then finds the two coefficients $A$ and $S$; the latter turns out to be

$$S = -e^{-2ika}\frac{(\sin ka + \frac{k}{u_0}\cos ka) + i\frac{k}{u_0}\sin ka}{(\sin ka + \frac{k}{u_0}\cos ka) - i\frac{k}{u_0}\sin ka}\,. \tag{A.18}$$

Eq. (A.18) shows that $S$ is a complex number with unit modulus: this property, called *unitarity*, ensures *flux conservation*: i.e., the incident and outgoing fluxes are equal [48–51]. As a consequence of unitarity, $S$ can be written as

$$S = e^{2i\delta}, \tag{A.19}$$

where the real quantity $\delta$ is known as the *phase shift*.

If $u_0 = 0$, (A.18) gives $S = -1$; if $u_0 \to \infty$, we find $S \to -e^{-2ika}$. Both limits are correct.

When $u_0 \to \infty$, there are bound states in the region $x \in [0, a]$ whenever $ka = n\pi$. When $u_0$ is large but finite, there is a small probability of "leaking" through the barrier and we observe a *resonance*. Resonances are seen as poles of the $S$-matrix in the complex $k$-plane, near $ka = n\pi$. Writing $ka = n\pi + \xi$, with $\xi \ll 1$, we find poles at

$$k_n a = k_n' a - i\gamma_n\,, \tag{A.20}$$

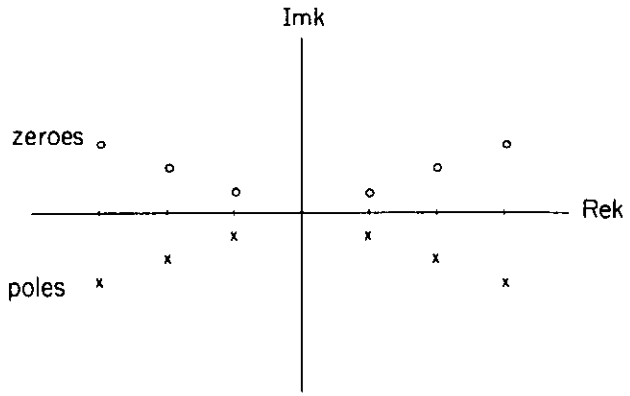


Fig. 12. Analytic structure of $S$ in the complex $k$ plane.

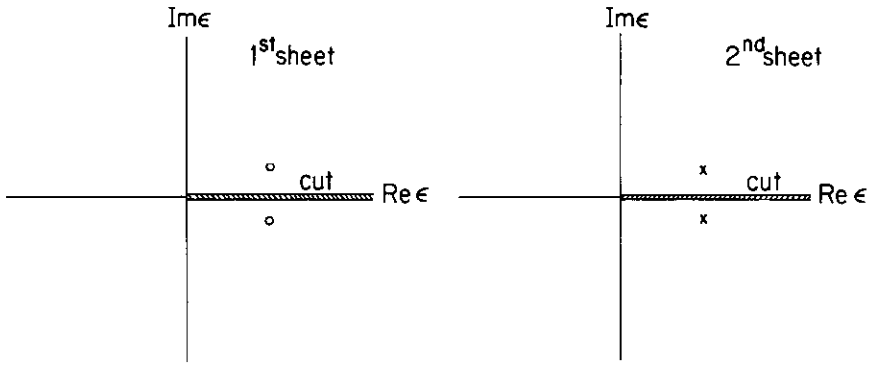


Fig. 13. Analytic structure of $S$ in the complex energy plane: the two Riemann sheets.

where

$$k_n' a = n\pi - \frac{n\pi}{u_0 a} + \frac{n\pi}{(u_0 a)^2} - \cdots \tag{A.21}$$

$$\gamma_n = \left(\frac{n\pi}{u_0 a}\right)^2 + \cdots . \tag{A.22}$$

Thus $S$ has poles in the lower half of the complex $k$-plane (*causality*) and zeroes at complex conjugate positions, to ensure unitarity (see fig. 12, where poles are indicated as crosses and zeroes as circles). In the energy variable $\epsilon = (ka)^2$ we need two Riemann sheets [48] and the resulting analytic structure is indicated schematically in fig. 13. The region shown in fig. 8b actually represents the lower



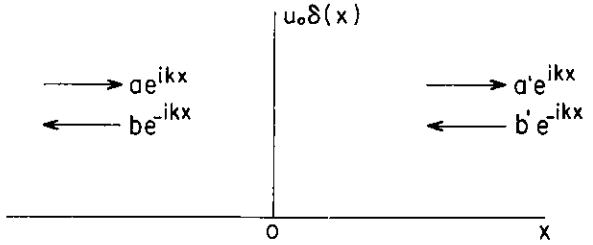


Fig. 14. One-dimensional example giving rise to a $2 \times 2$ scattering and transfer matrices.

part of the second sheet shown in fig. 13, continued analytically to the upper part of the first sheet.

*Appendix A.2.2. A $2 \times 2$ scattering matrix and the corresponding transfer matrix* In the present one-dimensional example we consider the spatial coordinate $x$ in the full interval $(-\infty, \infty)$ and study the scattering produced by a $\delta$-potential centered at $x = 0$ (see fig. 14). Continuity of the wave function and discontinuity of its slope at $x = 0$ yield

$$a' + b' = a + b \tag{A.23}$$

$$ik(a' - b') = ik(a - b) + u_0(a + b). \tag{A.24}$$

*1. The scattering matrix S.* If we decide to relate incoming and outgoing waves, we are led to the $2 \times 2$ scattering matrix $S$ ([51], p.96); i.e.

$$\begin{bmatrix} b \\ a' \end{bmatrix} = S \begin{bmatrix} a \\ b' \end{bmatrix}. \tag{A.25}$$

For instance, if $a = 1$, $b' = 0$, we have incidence from the left and $b = r$, $a' = t$ are the reflection and transmission amplitudes; if $a = 0$, $b' = 1$, we have incidence from the right and $b = t'$, $a' = r'$ are the corresponding reflection and transmission amplitudes. Thus the $S$ of (A.25) has the structure

$$S = \begin{bmatrix} r & t' \\ t & r' \end{bmatrix}. \tag{A.26}$$

Eqs. (A.25) and (A.26) are illustrations of the general relations (3.5) and (3.6) used in the text.

From Eqs. (A.24) we now obtain

$$\begin{bmatrix} 1 & -1 \\ 1 - \frac{u_0}{ik} & 1 \end{bmatrix} \begin{bmatrix} b \\ a' \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 + \frac{u_0}{ik} & 1 \end{bmatrix} \begin{bmatrix} a \\ b' \end{bmatrix}, \tag{A.27}$$

so that

$$S = \begin{bmatrix} \frac{\frac{u_0}{2ik}}{1-\frac{u_0}{2ik}} & \frac{1}{1-\frac{u_0}{2ik}} \\ \frac{1}{1-\frac{u_0}{2ik}} & \frac{\frac{u_0}{2ik}}{1-\frac{u_0}{2ik}} \end{bmatrix}. \tag{A.28}$$

We see that $S$ is a *unitary* matrix

$$SS^{\dagger} = I, \tag{A.29}$$

a property that again ensures flux conservation, just as in the $1 \times 1$ case treated above. We also see that $S$ is *symmetric*, $S = S^T$, as a consequence of time-reversal invariance [48–51].

2. *The transfer matrix* $M$. If we decide to relate the components on the two sides of the scatterer we are led to the transfer matrix $M$ (ref. [51], p.96); i.e.

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = M \begin{bmatrix} a \\ b \end{bmatrix}. \tag{A.30}$$

This equation is a particular case of the general relation (3.58). From the particular case $a = 1,\ b = r,\ a' = t,\ b' = 0$ one finds

$$r = -\beta^*/\alpha^* \tag{A.31}$$

$$t = 1/\alpha^*. \tag{A.32}$$

From $a = 0,\ b = t',\ a' = r',\ b' = 1$ one finds

$$t' = 1/\alpha^* \tag{A.33}$$

$$r' = \beta/\alpha^*. \tag{A.34}$$

We thus have the relation between the transfer matrix and the scattering matrix.

From (A.23), (A.24) we have

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1+\frac{u_0}{ik} & -1+\frac{u_0}{ik} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}, \tag{A.35}$$

so that

$$M = \begin{bmatrix} 1+\frac{u_0}{2ik} & \frac{u_0}{2ik} \\ -\frac{u_0}{2ik} & 1-\frac{u_0}{2ik} \end{bmatrix}. \tag{A.36}$$

We observe that $M$ of (A.36) has the structure of Eq. (3.62), so that (3.61) is satisfied. Eq. (3.59) which, in the present case, implies $|\alpha^2| - |\beta|^2 = 1$, is also fulfilled by (A.36).



The transfer matrix has the property of *multiplicativity*. Suppose two nonoverlapping scatterers, with transfer matrices $M_1$ and $M_2$; $M_1$ takes the coefficients $a, b$ to $a', b'$ and $M_2$ from $a', b'$ to $a'', b''$; it is clear that

$$M = M_2 M_1 \tag{A.37}$$

takes $a, b$ to $a'', b''$.

## Appendix B. Verification of the central-limit theorem for the $1 \times 1$ $K$ matrix

Set $E = 0$ in Eq. (3.42). Then

$$K - \sum_{i=1}^{n} \frac{\gamma_i^2}{E_i} = \sum_{i=1}^{n} \gamma_i^2 u_i. \tag{B.1}$$

Choose all the $\gamma_i^2$ and the $E_i$ 's statistically independent; the distribution $p(E_i)$ is uniform in the interval $(-L, -\delta), (\delta, L)$. The resulting probability density $q(u)$ for the variable $u = 1/E$ is then

$$q(u) = \frac{p(E = 1/u)}{u^2}. \tag{B.2}$$

The second moment of the variable $u$ is $\langle u^2 \rangle = \frac{1}{L\delta}$. As $\delta \to 0$, $\langle u^2 \rangle \to \infty$; thus $q(u)$ does *not* satisfy the requirements of the standard central-limit theorem. Therefore, we calculate directly the distribution of the variable $K$. We first compute the Fourier transform of $q(u)$; i.e.

$$f_u(k) = \langle e^{iku} \rangle = \frac{1}{L-\delta} \int_{1/L}^{1/\delta} \frac{\cos ku}{u^2} du, \tag{B.3}$$

an even function of $k$. We shall find it convenient to differentiate (B.3) to find

$$f_u'(k) = \frac{1}{L-\delta} [\mathrm{Si}(k/L) - \mathrm{Si}(k/\delta)], \tag{B.4}$$

an odd function of $k$. Here, $\mathrm{Si}(x)$ is the sine-integral function (ref.[101], p. 231). As a check, we expand (B.3) for small $k$; we find

$$f_u(k) = 1 - \frac{1}{2!}\frac{k^2}{L\delta} + \cdots. \tag{B.5}$$

We also expand (B.4), using

$$\mathrm{Si}(x) = x + \cdots, \quad x > 0, \tag{B.6}$$

to find

$$f_u'(k) = -\frac{k}{L\delta} + \cdots, \tag{B.7}$$

consistent with (B.5). The two sine-integral functions occurring in (B.4) are shown schematically in fig. 15, for $\delta \ll L$.

If we let $\delta \to 0$ in $(B.4)$, $\mathrm{Si}(k/\delta) \to \pi/2\mathrm{sgn}(k)$ and we find , for large $L$

$$f_u'(k) \to \frac{1}{L}[\frac{k}{L} - \frac{\pi}{2}\mathrm{sgn}(k)\cdots] \tag{B.8}$$

and, upon integrating

$$f_u(k) = 1 - \frac{\pi|k|}{2L} + \frac{k^2}{2L^2} + \cdots. \tag{B.9}$$

The Fourier transform of the distribution of $\gamma^2 u$ is thus

$$f_{\gamma^2 u}(k) = 1 - \frac{\pi\langle\gamma^2\rangle}{2L}|k| + \frac{\langle\gamma^4\rangle k^2}{2L^2} + \cdots \tag{B.10}$$

and that of the variable $K$, eq. (B.1), is

$$f_K(k) = [f_{\gamma^2 u}(k)]^n = \mathrm{e}^{n \ln f_{\gamma^2 u}(k)}$$

$$= \exp\left\{n\left[-\frac{\pi\langle\gamma^2\rangle}{2L}|k| + \frac{\langle\gamma^4\rangle k^2}{2L^2} - \frac{1}{2}\left(\frac{\pi\langle\gamma^2\rangle}{2L}k\right)^2 + \cdots\right]\right\}. \tag{B.11}$$

Now, taking the limit

$$n \to \infty, \quad L \to \infty, \quad 2L/n = \Delta, \tag{B.12}$$

we find

$$f_K(k) \to \mathrm{e}^{-\frac{\pi\langle\gamma^2\rangle}{\Delta}|k|}. \tag{B.13}$$

The inverse Fourier transform of (B.13) gives the probability density of the variable $K$ as

$$w(K) = \frac{t/\pi}{K^2 + t^2}, \tag{B.14}$$

where

$$t = \frac{\pi\langle\gamma^2\rangle}{\Delta}. \tag{B.15}$$

Notice that only the ratio $\langle\gamma^2\rangle/\Delta$ enters. We thus find *Cauchy's distribution in the limit.*

We now look for the distribution of $K$ derived from Poisson's kernel (3.36). If $S = \exp(i\theta)$, the variable $K$ of (3.41) is given by $K = \tan(\theta/2)$ and its



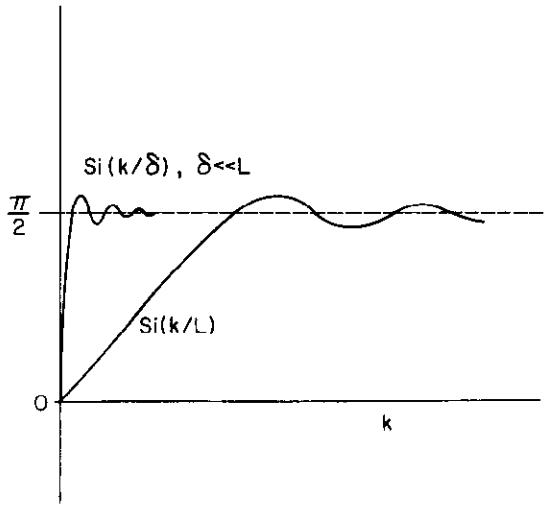


Fig. 15. The sine integral functions used in the text.

probability density by $w(K) = p(\theta(K))(d\theta/dK)$. Using (3.36) for $p(\theta)$ and also[69]

$$\overline{S} = \frac{1-t}{1+t}, \tag{B.16}$$

we obtain precisely (B.14).

Thus this calculation verifies, in a particular case (Levy's theorem [72,73]), the novel central-limit theorem referred to in the text, after Eq. (3.42).

## Appendix C. The Fokker–Planck equation for disordered systems

From Eqs. (3.65), (3.72) we have

$$M_{L+\delta L} = M_L + \delta M_L, \tag{C.1}$$

with

$$\delta M_L - \epsilon M_L. \tag{C.2}$$

Given a function $F(M)$ we can then write

$$F(M_{L+\delta L}) = F(M_L + \delta M_L) = \mathcal{D}_\epsilon(M_L)F(M_L), \tag{C.3}$$

where

$$\mathcal{D}_\epsilon(M) = 1 + (\delta M)\cdot\frac{\partial}{\partial M\cdot} + \frac{1}{2!}(\delta M)\cdot(\delta M)\cdots\frac{\partial^2}{\partial M\cdot\partial M\cdots} + \cdots, \tag{C.4}$$

$\delta M$ being given by (C.2); the dots in (C.4) indicate a contraction over indices. Averaging (C.3) we obtain

$$\langle F(M)\rangle_{L+\delta L} = \langle \mathcal{D}(M)F(M)\rangle_L\,, \tag{C.5}$$

where the differential operator $\mathcal{D}(M)$ is the average over $\epsilon$ (defined for a slice of thickness $\delta L$) of $\mathcal{D}_\epsilon(M)$ of (C.4); i.e.

$$\mathcal{D}(M) = 1 + \langle (\delta M)^{\cdot}\rangle^{(\epsilon)}_{\delta L}\frac{\partial}{\partial M^{\cdot}} + \frac{1}{2!}\langle (\delta M)^{\cdot}(\delta M)^{\cdots}\rangle^{(\epsilon)}_{\delta L}\frac{\partial^2}{\partial M^{\cdot}\partial M^{\cdots}} + \cdots . \tag{C.6}$$

Substituting (C.6) in (C.5) we find

$$\langle F(M)\rangle_{L+\delta L} = \langle F(M)\rangle_L + \left\langle \langle (\epsilon M)^{\cdot}\rangle^{(\epsilon)}_{\delta L}\frac{\partial F}{\partial M^{\cdot}} + \frac{1}{2!}\langle (\epsilon M)^{\cdot}(\epsilon M)^{\cdots}\rangle^{(\epsilon)}_{\delta L}\frac{\partial^2 F}{\partial M^{\cdot}\partial M^{\cdots}} + \cdots \right\rangle^{(M)}_L . \tag{C.7}$$

Using (3.73) and (3.74) we then obtain (3.75).

For $M_{\delta L}$ we use the parametrization of ref. [102]

$$M_{\delta L} = \begin{bmatrix} \mathrm{e}^{\theta} & 0 \\ 0 & \mathrm{e}^{\theta^*} \end{bmatrix} \begin{bmatrix} (1+\eta\eta^*)^{1/2} & \eta \\ \eta^* & (1+\eta^*\eta)^{1/2} \end{bmatrix}, \tag{C.8}$$

where $\theta = ih$, $h$ being $N\times N$ Hermitean and $\eta$, $N\times N$ complex symmetric. We choose the following averages of the first and second powers:

$$\langle\theta\rangle = \langle\eta\rangle = 0 \tag{C.9}$$

$$\langle\theta^2\rangle = -\langle\eta\eta^*\rangle \tag{C.10}$$

Eq. (3.73) is then satisfied to order $\delta L$. The terms linear in $\delta L$ in Eq. (3.74) are chosen as

$$\sigma^{11,11}_{ab,cd}\delta L = \langle\theta_{ab}\theta_{cd}\rangle = -\langle h_{ab}h_{cd}\rangle \equiv -\delta_{ad}\delta_{bc}\sigma'_{ab}\delta L \tag{C.11}$$

$$\sigma^{22,22}_{ab,cd}\delta L = \langle\theta^*\theta^*\rangle = -\langle h^*_{ab}h^*_{cd}\rangle \equiv -\delta_{ad}\delta_{bc}\sigma'_{ab}\delta L. \tag{C.12}$$

$$\sigma^{11,22}_{ab,cd} = \langle\theta_{ab}\theta^*_{cd}\rangle = \langle h_{ab}h^*_{cd}\rangle \equiv \delta_{ac}\delta_{bd}\sigma'_{ab}\delta L \tag{C.13}$$

$$\sigma^{12,21}\delta L = \langle\eta_{ab}\eta^*_{cd}\rangle \equiv \frac{\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}}{1+\delta_{ab}}\sigma_{ab}\delta L \tag{C.14}$$

$$\sigma^{11,12}\delta L = \langle\theta\eta\rangle = 0, \quad \sigma^{11,21}\delta L = \langle\theta\eta^*\rangle = 0 \tag{C.15}$$

$$\sigma^{12,12}\delta L = \langle\eta\eta\rangle \equiv 0, \quad \sigma^{21,21}\delta L = \langle\eta^*\eta^*\rangle \equiv 0 \tag{C.16}$$

$$\sigma^{12,22}\delta L = \langle\eta\theta^*\rangle \equiv 0, \quad \sigma^{21,22}\delta L = \langle\eta^*\theta^*\rangle \equiv 0 \tag{C.17}$$



Substituting in (3.75) we then obtain

$$\frac{\partial\langle F(M)\rangle_L}{\partial L} = \langle H(M)F(M)\rangle_L\,, \tag{C.18}$$

where

$$\begin{aligned} HF = &\sum_{abcdkm}\sigma'_{ab}\left\{M^{1k}_{bc}M^{2m}_{bd}\frac{\partial^2 F}{\partial M^{1k}_{ac}\partial M^{2m}_{ad}}\right\} \\ &-\frac{1}{2}\sum_{abcdkm}\sigma'_{ab}\left\{M^{1k}_{bc}M^{1m}_{ad}\frac{\partial^2 F}{\partial M^{1k}_{ac}\partial M^{1m}_{bd}} + M^{2k}_{bc}M^{2m}_{ad}\frac{\partial^2 F}{\partial M^{2k}_{ac}\partial M^{2m}_{bd}}\right\} \\ &+\sum_{abcdkm}\sigma_{ab}\left\{M^{1k}_{bc}M^{2m}_{bd}\frac{\partial^2 F}{\partial M^{2k}_{ac}\partial M^{1m}_{ad}} + M^{1k}_{ac}M^{2m}_{bd}\frac{\partial^2 F}{\partial M^{2k}_{bc}\partial M^{1m}_{ad}}\right\} \\ &-\sum_{acdkm}\sigma_{aa}M^{1k}_{ac}M^{2m}_{ad}\frac{\partial^2 F}{\partial M^{2k}_{ac}\partial M^{1m}_{ad}}. \end{aligned} \tag{C.19}$$

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