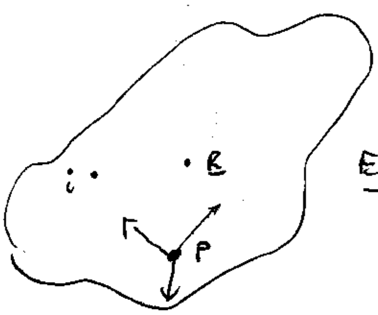
Merev testek mozgása :

tömegpontrendszer



$$|\underline{r}_i - \underline{r}_j|^2 = \text{cst} \quad \Rightarrow \quad 6 \text{ koordináta (szab. fok)}$$

Euler: mozgása leírható tetsz. kiv. P pontjának $\underline{r}_P(t)$ pályája + e körüli rotáció segítségével

$$\Rightarrow \boxed{\dot{\underline{r}}_i = \dot{\underline{r}}_P + \underline{\omega} \times (\underline{r}_i - \underline{r}_P)}$$

biz.: $0 = \dot{\underline{x}}_{iP} = B^T(\dot{\underline{x}}_{iP} - \underline{\omega} \times \underline{x}_{iP})$ $\left[\text{v. } \emptyset = \frac{d'\underline{r}_{iP}}{dt} = \dot{\underline{r}}_{iP} - \underline{\omega} \times \underline{r}_{iP}\right]$ ✓

együttmozgó koord $\quad \underline{r}_i - \underline{r}_P$ □

Áll.: $\underline{\omega}$ független P-től! $\left(\dot{\underline{r}}_Q = \dot{\underline{r}}_P + \underline{\omega} \times (\underline{r}_Q - \underline{r}_P) \ldots\right)$

$$\underline{r}_i = \tilde{\underline{r}}_i + \underset{\text{tkp.}}{\underline{R}} \qquad \underline{L} = ?$$

$$\underline{L} = \sum_{i=1}^{N} m_i \underline{r}_i \times \dot{\underline{r}}_i = \underbrace{M \underline{R} \times \dot{\underline{R}}}_{\underline{L}_{\text{pálya}}} + \underbrace{\sum_{i=1}^{N} m_i \tilde{\underline{r}}_i \times \underbrace{\dot{\tilde{\underline{r}}}_i}_{\underline{\omega} \times \tilde{\underline{r}}_i}}_{\underline{L}_{\text{belső}} = \underline{L}_{\text{tkp}}}$$

$$\underline{L}_{\text{tkp}} = \sum_i m_i \underbrace{\tilde{\underline{r}}_i \times (\underline{\omega} \times \tilde{\underline{r}}_i)} = \left(\sum_i m_i (\tilde{\underline{r}}_i^2 \mathbb{1} - \tilde{\underline{r}}_i \circ \tilde{\underline{r}}_i)\right) \cdot \underline{\omega}$$

$$\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}(\underline{a} \cdot \underline{c}) - \underline{c}(\underline{a} \cdot \underline{b}) \quad \Rightarrow \quad \underline{\omega} \cdot \tilde{\underline{r}}_i^2 - \tilde{\underline{r}}_i (\tilde{\underline{r}}_i \cdot \underline{\omega})$$

$$= (\tilde{\underline{r}}_i^2 \mathbb{1} - \tilde{\underline{r}}_i \circ \tilde{\underline{r}}_i) \cdot \underline{\omega}$$

komponensenként

$$L^{\alpha}_{\text{tkp}} = \sum_{\beta} \underbrace{\left\{ \sum_i m_i (\tilde{r}_i^2 \delta_{\alpha\beta} - \tilde{x}_i^{\alpha} \tilde{x}_i^{\beta}) \right\}}_{\Theta^{\alpha\beta}_{\text{tkp}}} \omega^{\beta}$$

tömegközépponti von. teh. nyom. tenzor

$$\boxed{L^{\alpha}_{\text{tkp}} = \Theta^{\alpha\beta}_{\text{tkp}} \omega^{\beta}} \quad \text{v.} \quad \boxed{\underline{L}_{\text{tkp}} = \Theta_{\text{tkp}} \underline{\omega}}$$

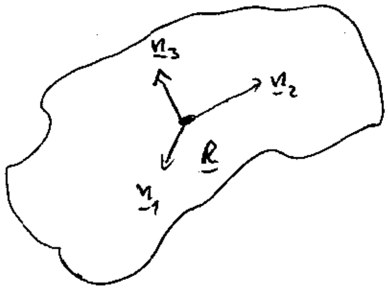
$$\boxed{\Theta^{\alpha\beta}_{tkp} = \sum_i m_i \left\{ (\underline{r}_i - \underline{R})^2 \delta^{\alpha\beta} - (x_i^\alpha - R_i^\alpha)(x_i^\beta - R_i^\beta) \right\}}$$

$$\Theta \equiv \Theta_{tkp} = \sum_i \begin{pmatrix} m_i(\tilde{y}_i^2 + \tilde{z}_i^2) & -m_i \tilde{x}_i \tilde{y}_i & -m_i \tilde{x}_i \tilde{z}_i \\ & m_i(\tilde{x}_i^2 + \tilde{z}_i^2) & -m_i \tilde{y}_i \tilde{z}_i \\ & & m_i(\tilde{x}_i^2 + \tilde{y}_i^2) \end{pmatrix}$$

- Θ valós szimmetrikus, poz. definit [$\forall\ \underline{n}$-re $\underline{n}\Theta\underline{n} > \emptyset$]

$\Rightarrow$ 3 poz. sajátérték, $\Theta_1, \Theta_2, \Theta_3$

3 egymásra mer. főtengely $\underline{n}_1, \underline{n}_2, \underline{n}_3$



$$\underline{L} = \underline{n}_1 \cdot \Theta_1(\underline{\omega} \cdot \underline{n}_1) + \underline{n}_2\, \Theta_2(\underline{\omega}\, \underline{n}_2) + \dots$$

ha $\underline{\omega} \parallel \underline{n}_i \Rightarrow \underline{L}_i = \underline{n}_i\, \Theta_i \omega$

Kinetikus energia:

$$K = \frac{1}{2} M \dot{\underline{R}}^2 + E_b$$

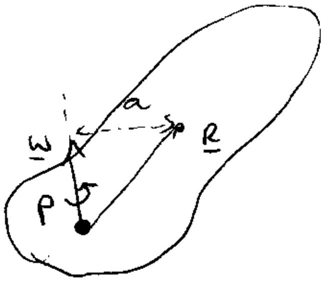
$$E_{belső} = \frac{1}{2} \sum_i m_i \dot{\tilde{\underline{r}}}_i^2 = \frac{1}{2} \sum_i m_i \underbrace{(\underline{\omega} \times \tilde{\underline{r}}_i)(\underline{\omega} \times \tilde{\underline{r}}_i)} =$$

$$(\underline{\omega} \times \underline{r}_i)(\underline{\omega} \times \underline{r}_i) = \underline{\omega} \cdot (\underline{r}_i \times (\underline{\omega} \times \underline{r}_i)) = \underline{\omega} \cdot (\underline{\omega}\, r_i^2 - \underline{r}_i (\underline{\omega} \cdot \underline{r}_i))$$

$$= \frac{1}{2} \sum_i \underline{\omega} \cdot \left[m_i (r_i^2 \mathbb{1} - \underline{r}_i \circ \underline{r}_i) \cdot \underline{\omega} \right] = \frac{1}{2} \underline{\omega}\, \Theta_{tkp} \underline{\omega}$$

$$\boxed{K = \frac{1}{2} M \dot{\underline{R}}^2 + \frac{1}{2} \underline{\omega}\, \Theta_{tkp}\, \underline{\omega}}$$

Rögz. pont körüli mozgás



P origó : $\dot{\underline{r}}_i = \underline{\omega} \times \underline{r}_i$

$$\Rightarrow \boxed{\underline{L} = \sum_i m_i \, \underline{r}_i \times \dot{\underline{r}}_i = \Theta_P \, \underline{\omega}}$$

$$\boxed{\Theta_P^{\alpha\beta} = \sum_i m_i \left(r_i^2 \delta^{\alpha\beta} - x_i^\alpha x_i^\beta \right)} \Rightarrow \int \text{alak}$$

hasonlóan : $\boxed{E_{\text{forgás}} = \frac{1}{2} \underline{\omega}^T \Theta_P \, \underline{\omega}}$

másfelől:

$$\underline{L} = \underline{L}_{\text{pálya}} + \underline{L}_{\substack{\text{tkp} \\ \text{(belső)}}} = M \, \underline{R} \times \underbrace{\dot{\underline{R}}}_{\underline{\omega} \times \underline{R}} + \Theta_{\text{tkp}} \, \underline{\omega} =$$

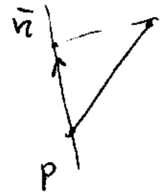
$$= \left(M \left(R^2 - \underline{R} \circ \underline{R} \right) + \Theta_{\text{tkp}} \right) \underline{\omega}$$

$$\boxed{\Theta_P^{\alpha\beta} = M \left(R^2 \delta^{\alpha\beta} - R^\alpha R^\beta \right) + \Theta_{\text{tkp}}^{\alpha\beta}}$$ ~ Steiner tétel

rögz. tengely körüli forgás $\underline{\omega} = \underline{n} \cdot \omega$; $\underline{n}^2 = 1$

$$E = \frac{1}{2} \underline{\omega} \, \Theta_P \, \underline{\omega} = \frac{1}{2} \omega^2 \left(\underline{n} \, \Theta_P \, \underline{n} \right)$$

$$I = \sum_i m_i \Big(\underbrace{r_i^2 - (\underline{n} \, \underline{r}_i)^2}_{l_i^2} \Big)$$ ← távolság tengelytől



fenti egyenletből $\Rightarrow$ $\underline{I = M \cdot a^2 + I_{\text{tkp}}} \geq I_{\text{tkp}}$

mozgásegyenlet:

$$\dot{\underline{L}} = \underline{N} \quad \sim \quad \boxed{\frac{d}{dt} \left(\Theta \, \underline{\omega} \right) = \underline{N}}$$

Θ forog!

spec. eset $\underline{n} = \text{cst} \parallel \hat{z}$

$$N_x = \frac{d}{dt}\left(\Theta_{xz}\,\omega\right)$$

$$N_y = \frac{d}{dt}\left(\Theta_{yz}\,\omega\right)$$

$$N_z = \frac{d}{dt}\left(\Theta_{zz}\,\omega\right)$$

$$\uparrow$$

$$\sum_i m_i\left(x_i^2 + y_i^2\right) = \text{cst}$$

még ha $N_z = 0$, akkor is $\omega = \text{cst}$ de

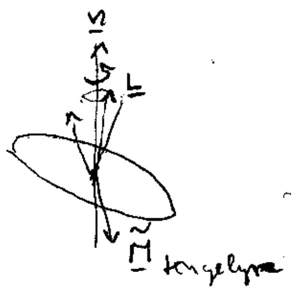
$$\frac{d}{dt}\Theta_{xz} = \sum_i m_i\,\dot{x}_i\,z_i = -\sum_i m_i\,\omega\,y_i z_i =$$

$$= -\omega\,\Theta_{yz}$$

$$\dot{\Theta}_{yz} = \omega\,\Theta_{xz}$$

$$\Rightarrow N_x = -\omega^2\,\Theta_{yz} \quad ; \quad N_y = +\omega^2\,\Theta_{xz} \quad !$$

üt a kerék ...



$\Theta_{xz} \sim \text{cst}\ \cos(\omega t)$

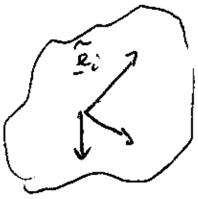
$\Theta_{yz} \sim \text{cst}\ \sin\omega t$

Szabad test mozgása: Euler egyenletek

tömegközéppont egyenletesen mozog: $R \to 0$

$$\tilde{L} \equiv B^T L$$

sebesség: $\dot{x} = \omega \times x \Rightarrow \tilde{\dot{x}} = B^T \dot{x} = B^T\omega \times B^T x = \tilde{\omega} \times \tilde{x}$



$$B^T L = B^T \sum_i m_i\, x_i \times \dot{x}_i = \sum_i m_i\, B^T x_i \times B^T \dot{x}_i$$

$$= \sum_i m_i\, B^T x_i \times \underbrace{B^T(\omega \times x_i)}_{B^T\omega \times B^T x_i} =$$

$$= \sum_i m_i\, \tilde{x}_i \times (\tilde{\omega} \times \tilde{x}_i) = \underbrace{\sum_i m_i (\tilde{x}_i^2 - \tilde{x}_i \circ \tilde{x}_i)}_{\tilde{\Theta}}\, \tilde{\omega}$$

$$\boxed{\tilde{L} = B^T L = \tilde{\Theta}\tilde{\omega}}\ ; \quad \tilde{\Theta} = \sum m_i (\tilde{x}_i^2 - \tilde{x}_i \circ \tilde{x}_i)$$

$$(\tilde{\Theta} = B^T \Theta B)$$

egyfelől: $\dot{\tilde{L}} = B^T(\underbrace{\dot{L}}_{= N} - \omega \times L) = \tilde{N} - \tilde{\omega} \times \tilde{L}$

szabad mozgás: $\boxed{\dot{\tilde{L}} = -\tilde{\omega} \times \tilde{L}}$

másfelől: $\dot{\tilde{L}} = \frac{d}{dt}(\tilde{\Theta}\tilde{\omega}) = \tilde{\Theta}\dot{\tilde{\omega}} \Rightarrow \boxed{\tilde{\Theta}\dot{\tilde{\omega}} = -\tilde{\omega} \times \tilde{\Theta}\tilde{\omega}\ (+\tilde{N})}$

$\uparrow$
$\tilde{\Theta} = cst$!

Euler egyenlet

célszerű koordinátarendszer: főtengely:

$\tilde{\omega} \equiv \Omega$

$$\boxed{\begin{aligned} \dot{\Omega}_1 I_1 &= (I_2 - I_3)\,\Omega_2 \Omega_3 \\ \dot{\Omega}_2 I_2 &= (I_3 - I_1)\,\Omega_3 \Omega_1 \\ \dot{\Omega}_3 I_3 &= (I_1 - I_2)\,\Omega_1 \Omega_2 \end{aligned}}$$

mozgásállandók:

(1) $E = \frac{1}{2}\Omega \tilde{\Theta} \Omega = \frac{1}{2}(\Omega_1^2 I_1 + \Omega_2^2 I_2 + \Omega_3^2 I_3) = cst = \left(\frac{M_1^2}{I_1} + \dots \frac{M_3^2}{I_3}\right)\frac{1}{2}$

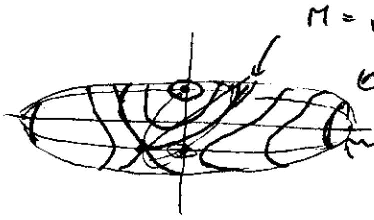
(2) $L^2 = \tilde{L}^2 = \Omega_1^2 I_1^2 + \dots = \boxed{M_1^2 + M_2^2 + M_3^2 = M^2 = cst}$

(1) $\Rightarrow$ ellipszoid:

$M = \sqrt{2EI_2}$

$I_1 > I_2 > I_3$

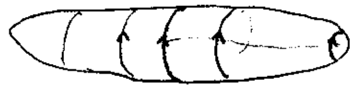
vonalak mentén mozog...



$\sqrt{I_1 2E}$

$\sqrt{2EI_3} \le M$

legegyszerűbb eset $I_1 = I_2 \neq I_3$



$I_3 < I_1$

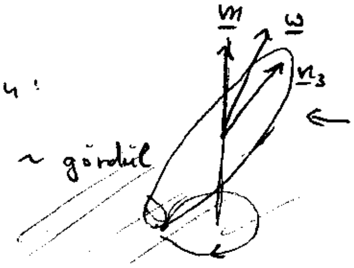
$\dot{\Omega}_3 = 0 \quad \Rightarrow \quad \Omega_3 = cst$

$$I_1 \dot{\Omega}_1 = \left(\Omega_3 (I_1 - I_3)\right) \Omega_2$$
$$I_1 \dot{\Omega}_2 = -(\;\ldots\;) \Omega_1$$

$$\Rightarrow \quad (\Omega_1 + i\Omega_2)^{\cdot} = -i \underbrace{\left(\frac{\Omega_3 (I_1 - I_3)}{I_1}\right)}_{\Omega} (\Omega_1 + i\Omega_2)$$

$$\Omega_1 = A \cos(\Omega t)$$
$$\Omega_2 = A \sin(\Omega t)$$

állítás: álló koordinátarendszerben:

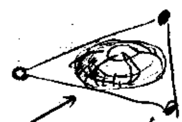


$\underline{u}\Theta\underline{u} = 1$ ellipszoid: tehetetlenségi ellipszoid

Poinsot tétele: $D = \{\underline{\Omega}\tilde{\Theta}\underline{\Omega} = 1\}$ ellipszoid

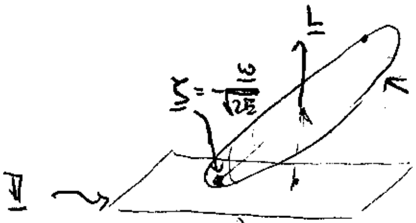
tehetetlenségi ellipsz ~ test

$$\operatorname{grad} D = 2\tilde{\Theta}\underline{\Omega} = 2\tilde{\underline{L}}$$



Ennek!

$\Rightarrow$ $\underline{L}$ merőleges az $\underline{\omega}\Theta\underline{\omega} = 1$ ellipszoid érintőjére az $\underline{\varsigma} = \pm \frac{\underline{\omega}}{\sqrt{2E}}$ pontban



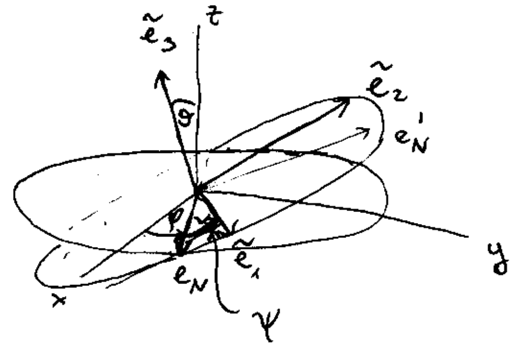
~ a síkra $\quad \underline{\varsigma} \cdot \underline{L} = -\frac{1}{\sqrt{2E}} \underline{\omega}\Theta\underline{\omega} = -\sqrt{2E} = cst$

$\Rightarrow$ a $-\frac{\underline{\omega}}{\sqrt{2E}}$ pont mindig ezen a síkon mozog!

rajta van a forgási tengelyen $\Rightarrow$ gördül

Euler szögek:

3 rotáció: $\cdot\, dt$: $\vartheta \to \vartheta + d\vartheta$ ~ rotáció $\underline{e}_N$ körül



$dt\, \underline{\omega}_\vartheta$ szöggel $\Rightarrow dt\, \underline{\omega}_\vartheta \times$

$$\Rightarrow \quad \underline{\omega}_\vartheta = \dot{\vartheta}\, \underline{e}_N$$

$$\underline{e}_N = \cos\psi\, \tilde{\underline{e}}_1 - \sin\psi\, \tilde{\underline{e}}_2$$

$$\underline{\omega}_\vartheta = \dot{\vartheta}\cos\psi\, \tilde{\underline{e}}_1 - \dot{\vartheta}\sin\psi\, \tilde{\underline{e}}_2$$

$\psi \to \psi + d\psi$ ~ rot. $\hat{\underline{e}}_3$ körül

$$\Rightarrow \underline{\omega}_\psi = \dot{\psi}\, \tilde{\underline{e}}_3$$

$\varphi \to \varphi + d\varphi$ ~ rotáció $\hat{z}$ körül

$$\Rightarrow \underline{\omega}_\varphi = \dot{\varphi}\, \hat{z} = \dot{\varphi}\left(\cos\vartheta\, \tilde{\underline{e}}_3 + \underbrace{\sin\vartheta\cos\psi\, \tilde{\underline{e}}_2 + \sin\vartheta\sin\psi\, \tilde{\underline{e}}_1}_{\sin\vartheta\, e_N'}\right)$$

így végezetül

$$\left(dt\, \underline{\omega}_\vartheta \times + dt\, \underline{\omega}_\varphi \times + dt\, \underline{\omega}_\psi \times\right)\underline{r} = dt\, \underline{\omega} \times \underline{r}$$

$$\underline{\omega} = \underline{\omega}_\vartheta + \underline{\omega}_\varphi + \underline{\omega}_\psi$$

$$\begin{aligned}\underline{\omega} = {} & \left(\dot{\vartheta}\cos\psi + \dot{\varphi}\sin\vartheta\sin\psi\right)\tilde{\underline{e}}_1 \\ & + \left(\dot{\varphi}\sin\vartheta\cos\psi - \dot{\vartheta}\sin\psi\right)\tilde{\underline{e}}_2 \\ & + \left(\dot{\psi} + \dot{\varphi}\cos\vartheta\right)\tilde{\underline{e}}_3\end{aligned}$$

$$\Rightarrow \boxed{\begin{aligned}\Omega_1 &= \dot{\vartheta}\cos\psi + \dot{\varphi}\sin\vartheta\sin\psi \\ \Omega_2 &= -\dot{\vartheta}\sin\psi + \dot{\varphi}\sin\vartheta\cos\psi \\ \Omega_3 &= \dot{\psi} + \dot{\varphi}\cos\vartheta\end{aligned}}$$

szimmetrikus pörgettyűre: $\Omega_1^2 + \Omega_2^2 = A^2 = \text{cst} = \dot{\vartheta}^2 + \dot{\varphi}^2\sin^2\vartheta$

$$\left.\begin{aligned} L_z = L \Rightarrow \tilde{L}_3 &= \underline{L}\cdot\tilde{\underline{e}}_3 = L\cos\vartheta = \text{cst} \\ \tilde{L}_3 &= I_3\,\underbrace{\Omega_3 = \text{cst}}_{\text{Euler}} \Rightarrow \end{aligned}\right\} \Rightarrow \vartheta = \text{cst}$$

$$\Rightarrow \dot{\varphi} = \pm\left(\frac{\sin\vartheta}{A}\right)^{-1} \Rightarrow \boxed{\varphi = t\,\frac{\sqrt{\Omega_1^2 + \Omega_2^2}}{\sin\vartheta} + \text{cst}}$$ rotáció $\hat{z}$ körül

$$\Rightarrow \boxed{\dot{\psi} = \Omega_3 - \text{ctg}\,\vartheta\sqrt{\Omega_1^2 + \Omega_2^2}}$$

némi számolás: $\text{tg}\,\vartheta = \dfrac{I_1\Omega_1}{I_3\Omega_3}$