Hamilton – formalizmus

Lagrange: $L(q,\dot q,t) \rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot q} = \frac{\partial L}{\partial q}$

Legendre transzformáció $\left(\begin{array}{l} E(V,N,S) \Rightarrow \frac{\partial E}{\partial S} = T \Rightarrow F(T,V,N) = \\ dE = -p\,dV + \mu\,dN + T\,dS \qquad = E(V,N,S(T,..)) - TS \end{array} \right)$

$\downarrow$

itt: $p \equiv \frac{\partial L}{\partial \dot q}(q,\dot q,t) \rightarrow$ invertálva $\dot q = \dot q(p,q,t)$

$$\boxed{H(q,p,t) \equiv p\cdot \dot q(p,q,t) - L(q,\dot q(q,p,t),t)}$$

$$\boxed{H = p\cdot\dot q - L}$$

- mozgásegyenletek:

$$\left(\frac{\partial H}{\partial q}\right)_{p,t} = \sum_e p_e \frac{\partial \dot q_e}{\partial q} - \left(\frac{\partial L}{\partial q}\right)_{q,\dot q} - \sum_e \left(\frac{\partial L}{\partial \dot q}\right)_e \frac{\partial \dot q_e}{\partial q} = -\left(\frac{\partial L}{\partial q}\right)_{q,\dot q} = -\frac{d}{dt}\frac{\partial L}{\partial \dot q}$$

$$= -\frac{dp}{dt} \Rightarrow \boxed{\dot p = -\frac{\partial H}{\partial q}}$$

$$\frac{\partial H}{\partial p} = \dot q + \sum_e p_e \frac{\partial \dot q_e}{\partial p} - \sum_e \underbrace{\frac{\partial L}{\partial \dot q_e}}_{p_e} \frac{\partial \dot q_e}{\partial p} = \dot q$$

$$\boxed{\dot q = \frac{\partial H}{\partial p}} \qquad \boxed{\dot p = -\frac{\partial H}{\partial q}}$$

elsőrendű differenciálegy.

példa: inga: $L = \frac{m}{2} l^2 \dot\varphi^2 - mgl(1-\cos\varphi) \Rightarrow \frac{ml^2\dot\varphi^2}{2} + mgl\cos\varphi$

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = ml^2\dot\varphi \Rightarrow \dot\varphi = \frac{p_\varphi}{ml^2}$$

$$H = p_\varphi\cdot\dot\varphi - L = \frac{p_\varphi^2}{ml^2} - \left(\frac{ml^2}{2}\frac{p_\varphi^2}{m^2l^4} + mgl\cos\varphi\right)$$

$$\boxed{H(p_\varphi,\varphi) = \frac{1}{2}\frac{p_\varphi^2}{ml^2} - mgl\cos\varphi}$$

$$\dot\varphi = \frac{\partial H}{\partial p_\varphi} = \frac{1}{ml^2}p_\varphi = \frac{p_\varphi}{\Theta} \qquad \left(\Rightarrow \ddot\varphi = -\frac{g}{l}\sin\varphi\right)$$

$$\dot p_\varphi = -\frac{\partial H}{\partial \varphi} = -mgl\sin\varphi$$

Láttuk: $\frac{dH}{dt} = 0$ ha $L(q, \dot q, \cancel{t})$

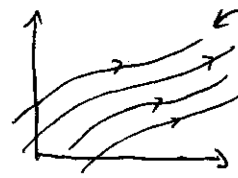
Hamilton egyenletekből:

$$\frac{dH}{dt} = \dot p \underbrace{\frac{\partial H}{\partial p}}_{\dot q} + \dot q \underbrace{\frac{\partial H}{\partial q}}_{-\dot p} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$$

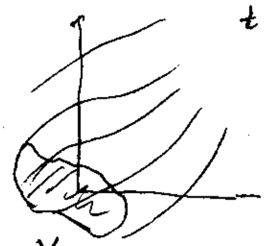
$$\boxed{\frac{dH}{dt} = \frac{\partial H}{\partial t}}$$

autonom rendszer

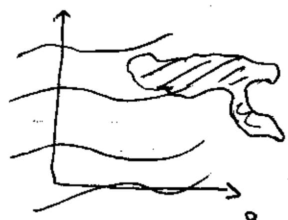
fázistér



Liouville: a Hamilton-rendszer folyadékként folyik

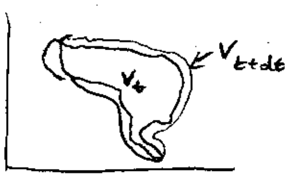


$\Rightarrow$



$$\boxed{V_t = V_0}$$ $\Rightarrow$ stat. fiz.

biz.: $V_t \to V_{t+dt}$



$$V_t = \int_{V_t} d^f q \, d^f p$$

dt: $p \to p + dt\,\dot p = p - dt \frac{\partial H}{\partial q} \equiv p'(q,p)$

$q \to q + dt \frac{\partial H}{\partial p} \equiv q'(q,p)$

$$V_{t+dt} = \int_{V_{t+dt}} d^f q' \, d^f p' = \int_{V_t} \left| \frac{\partial(q', p')}{\partial(q, p)} \right| d^f q \, d^f p$$

$$\det \begin{pmatrix} \frac{\partial q'}{\partial q} & \frac{\partial p'}{\partial q} \\ \frac{\partial q'}{\partial p} & \frac{\partial p'}{\partial p} \end{pmatrix} \doteq \det \left(\begin{array}{c|c} \delta_{lm} + dt \frac{\partial^2 H}{\partial q_m \partial p_l} & -dt \frac{\partial^2 H}{\partial q_m \partial q_l} \\ \hline dt \frac{\partial^2 H}{\partial p_m \partial p_l} & \delta_{ml} - \frac{\partial^2 H}{\partial p_m \partial q_l} dt \end{array} \right)$$

$$\frac{\partial q'_l}{\partial q_m} = \delta_{lm} + dt \frac{\partial^2 H}{\partial p_l \partial q_m}$$

$$\Rightarrow \det \begin{pmatrix} 1 + dt\ \partial_q \partial_p H & -dt\ \partial_q \partial_q H \\ dt\ \partial_p \partial_p H & 1 - dt\ \partial_p \partial_q H \end{pmatrix} = 1 + dt\ \mathrm{tr}\,\underbrace{\left(\partial_q \partial_p H - \partial_p \partial_q H\right)}_{=0} + \mathcal{O}(dt^2)$$

$$\det\left(1 + \epsilon \underline{\underline{A}}\right) = 1 + \epsilon\, \mathrm{tr}\, A + \dots$$

$$V_{t+dt} = V_t + \mathcal{O}(dt^2) \Rightarrow \boxed{\frac{dV_t}{dt} = 0} \quad \checkmark$$

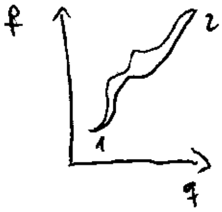
□

projekciók is megmaradnak ...

. **Módosított Ham.-elv:**

$$\boxed{S = \int_{t_1}^{t_2} dt \left\{ \dot q p - H(p,q) \right\}} \leftarrow S[p,q]$$

Áll: a megvalósuló utakra (pályákra) $\boxed{\delta S = 0 \Leftrightarrow \dot q = \frac{\partial H}{\partial p},\ \dot p = -\frac{\partial H}{\partial q}}$



δq, δp függetlenül megválasztható!

biz.:
$$\delta \int_{t_1}^{t_2} dt \left\{ \dot q(p,q,t) \cdot p - H(p,q,t) \right\} = S[p+\delta p, q+\delta q] - S[p,q]$$

$$\int_{t_1}^{t_2} \left(\delta \dot q \cdot p + \dot q \cdot \delta p - \frac{\partial H}{\partial p} \delta p - \frac{\partial H}{\partial q} \delta q \right) dt$$

$$= \underbrace{\left[\delta q \cdot p\right]_1^2}_{=0} + \int_{t_1}^{t_2} \left\{ \left(\dot q - \frac{\partial H}{\partial p} \right) \cdot \delta p - \left(\frac{\partial H}{\partial q} + \dot p \right) \cdot \delta q \right\} dt$$

$$\forall\ \delta p\text{-re}\quad \delta q\text{-ra} \quad \Rightarrow \quad \boxed{\begin{aligned} \dot q &= \frac{\partial H}{\partial p} \\ \dot p &= -\frac{\partial H}{\partial q} \end{aligned}}$$

visszafelé triviális. □

Fiz. mennyiségek időfejlődése:

pl. $\frac{dK}{dt} = ?$ $\qquad \frac{dL_z}{dt} = ?$

$F(p, q, t)$

$$\frac{dF}{dt} = \frac{\partial F}{\partial q}\cdot \dot q + \frac{\partial F}{\partial p}\dot p + \frac{\partial F}{\partial t} = \frac{\partial F}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial F}{\partial t}$$

Poisson zárójel: $\boxed{[F, G] \equiv \sum_{e=1}^{f}\left(\frac{\partial F}{\partial q_e}\frac{\partial G}{\partial p_e} - \frac{\partial F}{\partial p_e}\frac{\partial G}{\partial q_e}\right)}$

$$\Rightarrow \quad \boxed{\frac{dF}{dt} = [F, H] + \frac{\partial F}{\partial t}}$$

- F megmaradó, ha $\frac{dF}{dt} = 0$ $\qquad$ ha $F(p, q, \cancel{t}) \Rightarrow$
  $[F, H] = 0$

speciálisan $\quad F \to p_m \Rightarrow \quad \dot p_m = \sum_e \left(\underbrace{\frac{\partial p_m}{\partial q_e}}_{\emptyset}\frac{\partial H}{\partial p_e} - \underbrace{\frac{\partial p_m}{\partial p_e}}_{\delta_{me}}\frac{\partial H}{\partial q_e}\right) = -\frac{\partial H}{\partial q_m}$ ✓

Szimplektikus struktúra:

$$z \equiv \{q, p\} \quad ; \quad J \equiv \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{2f \times 2f}$$

így

$$\dot z = \begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = \begin{pmatrix} \partial_p H \\ -\partial_q H \end{pmatrix} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} \partial_q H \\ \partial_p H \end{pmatrix} = J\frac{\partial H}{\partial z}$$

$$\boxed{\dot z = J\frac{\partial H}{\partial z}}, \quad \text{azaz} \quad \boxed{\dot z_i = J_{ik}\frac{\partial H}{\partial z_k}}$$

J tulajd.: $\qquad J^T = -J \quad ; \quad J^2 = -1 \; ; \quad \det J = +1$

így $\qquad \frac{dF}{dt} = \frac{\partial F}{\partial z}\cdot \dot z + \frac{\partial F}{\partial t} = \frac{\partial F}{\partial z} J \frac{\partial H}{\partial z} + \frac{\partial F}{\partial t}$

$$\Rightarrow \quad \boxed{[F, G] = \frac{\partial F}{\partial z} J \frac{\partial G}{\partial z}}$$

Poisson – zárójelek tulajd.:

- $[F, G] = -[G, F]$
- $[F, G_1 + G_2] = [F, G_1] + [F, G_2]$
- $[F, G_1 \cdot G_2] = [F, G_1]\, G_2 + G_1 [F, G_2]$
- $[F_1, [F_2, F_3]] + [F_2, [F_3, F_1]] + [F_3, [F_1, F_2]] = \emptyset$

(Jacobi – azonosság)

$[\ ,\ ]$ művelet a fázis téren ért. fv-ek terén

$\sim$ Lie – algebra

- tulajdonságai $\sim$ mátrix szorzás
- Q.M.: $F, H \rightarrow \hat{F}, \hat{H}$

$$[F, H] \longrightarrow \frac{1}{\hbar i} [\hat{F}, \hat{H}]$$

- klasszikus mechanika megfogalmazható $[\ ,\ ]$-kkel
- kanonikus koordinátázástól független: $[F, G]_{p,q} = [F, G]_{\underline{P},\underline{Q}}$

Poisson – tétel: ha F_1 és F_2 mozgásállandó, akkor $[F_1, F_2]$ is, azaz

$$\frac{dF_1}{dt} = 0, \quad \frac{dF_2}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}[F_1, F_2] = 0$$

biz.: egyszerű eset $\frac{\partial F_1}{\partial t} = \frac{\partial F_2}{\partial t} = 0$

ekkor $\frac{d}{dt}[F_1, F_2] = [[F_1, F_2], H] = -[H, [F_1, F_2]] =$

$$= [F_1, \underbrace{[F_2, H]}_{\frac{dF_2}{dt} = 0}] + [F_2, \underbrace{[H, F_1]}_{-\frac{dF_1}{dt} = 0}] = \emptyset \quad ✓$$

ha $\frac{\partial F_{1,2}}{\partial t} \neq 0$

$$\frac{d}{dt}[F_1, F_2] = \underbrace{[[F_1, F_2], H]}_{\text{Jacobi}} + \underbrace{\frac{\partial}{\partial t}[F_1, F_2]}_{\left[\frac{\partial F_1}{\partial t}, F_2\right] + \left[F_1, \frac{\partial F_2}{\partial t}\right]} = \ldots = \left[\frac{dF_1}{dt}, F_2\right] + \left[F_1, \frac{dF_2}{dt}\right] = \emptyset$$

$\square$

Néhány Poisson-zárójel:

$$[q_e, q_m] = \sum_k \left(\frac{\partial q_e}{\partial q_k} \underbrace{\frac{\partial q_m}{\partial p_k}}_{\emptyset} - \dots \right) = 0$$

$$\Rightarrow \boxed{[\eta_i, \eta_j] = \mathcal{J}_{ij}}$$

$$[q_e, p_m] = \sum_k \left(\underset{\delta_{ek}}{\frac{\partial q_e}{\partial q_k}} \underset{\delta_{mk}}{\frac{\partial p_m}{\partial p_k}} - \dots \right) = \delta_{em}$$

$$L_i \equiv \epsilon_{ijk} x^j p^k \qquad L_z = x \cdot p_y - y \cdot p_x$$

$$[p_x, L_z] = [p_x, x p_y - y p_x] = [p_x, x p_y] - [p_x, y p_x] = -p_y$$

$$x \underset{\emptyset}{[p_x, p_y]} + \underbrace{[p_x, x]}_{-1} p_y$$

$$[p_y, L_z] = [p_y, x p_y - y p_x] = -[p_y, y] p_x = p_x$$

$$[x, L_z] = -y \; ; \quad [y, L_z] = -x \quad ;$$ H. F.: $[L_x, L_y] = L_z$

szimmetriák?

$$F(t+dt) = F(t) + dt\,[F, H] \Rightarrow dF = dt\,[F, H] + \dots$$

$$\delta\varphi \left[\begin{pmatrix} p_x \\ p_y \end{pmatrix}, L_z \right] = \begin{pmatrix} -p_y \delta\varphi \\ p_x \delta\varphi \end{pmatrix} = \delta p \Rightarrow \delta p = \delta\varphi [p, L_z]$$

$p + \delta p$ elforgatott rendszer impulzusa

- H időbeli eltolást generál ...
- L_z forgást z körül ...
- P_x eltolást x irányba ...

Látni fogjuk: Poisson-zárójel független q, p megválasztásától, amennyiben kanonikus koordinátákat használunk

szimmetriák :

láttuk $\qquad p_x \to \delta\varphi[p_x, L_z] = -\delta\varphi\, p_y = \delta p_x$

$$\delta\varphi[p_y, L_z] = \delta\varphi\, p_x = \delta p_y$$

hasonló egyenletek x, y-ra

$$F(\underline{x}, \underline{p}) \Rightarrow \delta F = F(\underline{x}+\delta\underline{x}, \underline{p}+\delta\underline{p}) - F(\underline{x}, \underline{p})$$

$\underline{x}' = \underline{x}+\delta\underline{x}$

$\underline{p}' = \underline{p}+\delta\underline{p}$ $\qquad = \delta\varphi[F, L_z]$ (*)

H forgásinvariáns $\qquad \delta H = \delta\varphi[H, L_z] = 0$

de ekkor $\qquad \frac{dL_z}{dt} = [L_z, H] = -[H, L_z] = 0$!

- L_z szimmetria "generátora" (most forgás), ha $\boxed{[H, L_z] = 0}$ $\Rightarrow$ ekkor $\boxed{\frac{dL_z}{dt} = 0}$

megmaradó mennyiségek szimmetriát generálnak!

kieg: (*) bizonyítása

$\underline{z} \equiv (\underline{x}, \underline{p}) \qquad \delta\underline{z} = \delta\varphi[\underline{z}, L_z] = \delta\varphi\, \mathcal{J} \frac{\partial L_z}{\partial \underline{z}}$

$$\Rightarrow \delta F = F(\underline{z}+\delta\underline{z}) - F(\underline{z}) \approx \frac{\partial F}{\partial \underline{z}} \cdot \delta\varphi[\underline{z}, L_z] = \delta\varphi \frac{\partial F}{\partial \underline{z}} \mathcal{J} \frac{\partial L_z}{\partial \underline{z}}$$

$$= \delta\varphi[F, L_z] \quad ✓$$

(H.F.: $\qquad P_x = \sum_i p_{ix}$ $\qquad$ x irányú eltolást generál!

azaz $\qquad F(\underset{\uparrow \atop \delta a \cdot \hat{x}}{\underline{x}+\delta\underline{x}}, \underline{p}) - F(\underline{x}, \underline{p}) = \delta a [F, P_x]$)