Folytonos közegek mechanikája:

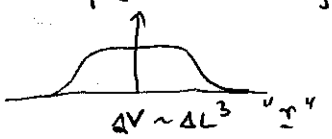
gáz v. folyadék .. $\sim 10^{23}$ szab. fok
reménytelennek tűnik ..

de TKP mozgás leválasztható

$\sim$ Föld mozgása

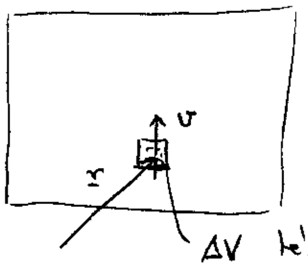
folyadék: $\hat{\varrho}(\underline{r},t) = \sum_i m_i\, \delta(\underline{r} - \underline{r}_i(t))$

átlagolás: $f(\underline{r})$: $\int d^3\underline{r}\, f(\underline{r}) = 1$



$$\Rightarrow \quad \varrho(\underline{r}) \equiv \int d\underline{r}'\, f(\underline{r}-\underline{r}')\, \hat{\varrho}(\underline{r}') = \sum_i m_i\, f(\underline{r}-\underline{r}_i(t))$$

sima fv. ha $\Delta L \gg a \sim$ atomok tip. távolsága



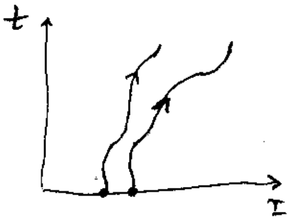
ΔV-ben $m_{\Delta V} \approx \varrho(\underline{r})\, \Delta V$ tömeg

$\langle |\underline{v}_i| \rangle \sim$ hangsebesség de közeli atomok együtt mozognak v. átlagsebessége $|\underline{v}| \ll |\underline{v}_i|$!

jól definiált $\rightarrow$ $\boxed{\underline{v}(\underline{r},t)}$ sebességmező

$\rightarrow$ $\boxed{\underline{r}'(t,\underline{r})}$ pályák ~ trajektóriák is jól definiáltak

$x_i' = x_i + s_i$ $\quad || \quad$ $\underline{v}(\underline{r},t) \overset{\text{vigyázat}}{\neq} \partial_t\, \underline{s}(\underline{r},t)$



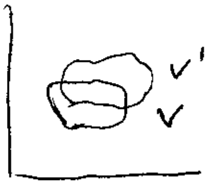
Teljes derivált vs. parciális derivált, kontinuitási egy.:

$\varrho(\underline{r},t)$ $\rightarrow$ pályamenti derivált: $d_t\varrho = \partial_t\varrho + \frac{\partial\varrho}{\partial\underline{r}}\cdot\underline{v}$

$$\boxed{d_t\varrho = \partial_t\varrho + v_i\,\partial_i\varrho}$$

integrált mennyiség változása:

$$m_V(t) \equiv \int_V d^3r \; \varrho(\underline{r},t)$$



$$\boxed{\partial_t m_V(t) = \int_V d^3r \; \partial_t \varrho(\underline{r},t)}$$

"teljes" derivált

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t}\left(m_{V'}(t'=t+\Delta t) - m_V(t)\right) \equiv D_t m_V$$

$$\underline{r}'(t) = \underline{r} + \Delta t \; \underline{v}(\underline{r},t) \longrightarrow x_i' = x_i + \Delta t \, v_i$$

$$m_{V'} = \int_{V'} d^3x' \; \varrho(x_i' = x_i + \Delta t v_i, \; t+\Delta t) =$$

$$\left|\frac{\partial x_i'}{\partial x_j}\right| d^3x \qquad \det\left(\delta_{ij} + \Delta t \, \partial_j v_i\right) \approx 1 + \Delta t \, \partial_i v_i$$

$$= \int_V d^3x \; (1 + \Delta t \; \partial_i v_i)(\varrho + \Delta t \partial_t \varrho + \Delta t \; v_i \partial_i \varrho) + \mathcal{O}(\Delta t^2)$$

$$= m_V + \Delta t \int_V d^3x \left(\partial_t \varrho + \underbrace{(\partial_i v_i)\varrho + v_i \partial_i \varrho}_{\partial_i (v_i \varrho)} \right)$$

$$\boxed{D_t m_V = \int_V d^3x \; (\partial_t \varrho + \partial_i (v_i \varrho))}$$

← bármilyen mennyiségre teljesül ...
$\varrho \to$ energiasűrűség
töltés ..

tömegmegmaradás: $D_t m_V = \emptyset!$

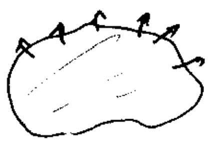
$$\Rightarrow \forall V\text{-re} \int_V \ldots = 0 \Longrightarrow \boxed{\partial_t \varrho + \operatorname{div} \underline{j} = 0}$$

$\underline{j} = \underline{v} \cdot \varrho$
kontinuitási egy.

értelmezések:

- Gauss-tétel:

$$\partial_t m_V = \int d^3x \; \partial_t \varrho = - \underbrace{\int d\underline{A} \; (\underline{v} \cdot \varrho)}_{\text{kiáramló anyag}}$$



– másik ért.:

$$\partial_t \varrho + \partial_i(v_i \varrho) = \underbrace{\partial_t \varrho + v_i \partial_i \varrho}_{d_t \varrho} + \varrho(\partial_i v_i) = 0$$

$\operatorname{div} \underline{v} \sim$ kitágulás
időegység alatti
rel. térfogatvált.

– ∀ megmaradó mennyiségre működik

– $\partial_t \varrho = -\operatorname{div} \underline{v} \cdot \varrho \Rightarrow \underline{v}(\underline{r},t),\ \varrho(\underline{r},t)$ ismert
$\Rightarrow \varrho(\underline{r}, t+\Delta t)$!

Erők és mozgásegy.:

ΔV  $d\underline{F} \sim dA$

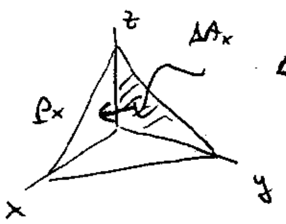
$\Delta \underline{F}_h$ hosszútávú $\Rightarrow \Delta \underline{F}_h = \int_{\Delta V} d^3\underline{r}\ f(\underline{r},t)$

- $f(\underline{r},t) = g \cdot \varrho$
  grav. mező

- $f(\underline{r},t) = \varrho_e \underline{E} + \varrho_e \underline{v} \times \underline{B}$
  $(\underline{r},t)$ fv.-e

rövidtávú felületi erők:

áll.: $\boxed{d\underline{F} = \underline{\underline{\sigma}}\, d\underline{A}}$ $\quad dF_i = \sigma_{ik}\, dA_k$, ahol
σ_{ik} szimmetrikus tenzor
feszültségtenzor

biz.: 

$\Delta \underline{A} \rightarrow \Delta A_x \rightarrow$ erre p_x erő hat m^2-enként (nyomás)

$$\Delta A\, p + p_x \Delta A_x + \dots + p_z \Delta A_z = (\varrho \Delta V)\, \underline{a}$$

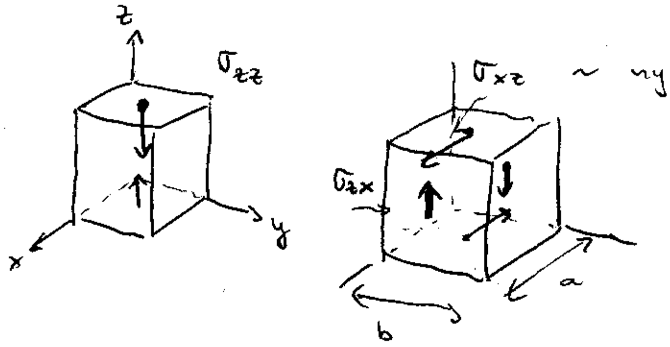
$$\Delta V \rightarrow \emptyset \Rightarrow p_x \Delta A_x + p_y \Delta A_y + \dots + p\, \Delta A = 0$$

$$\Delta \underline{F}_A = p \cdot \Delta A = -\sum_j p_j \Delta A_j \Rightarrow (\Delta F_A)_i = \underbrace{-p_j^i}_{\equiv\ \sigma_{ij}} \Delta A_j$$

$$\boxed{dF_i = \sigma_{ij}\, dA_j} \quad \boxed{p_i = \sigma_{ij}\, n_j} \leftarrow p_A = \underline{\underline{\sigma}}\, \underline{n}$$

$p(\underline{n})$

□



~ nyírás!

egyensúlyban (y-ra vett forg. ny.)

$$a \cdot (\sigma_{zx} bc) = c\,(ab\,\sigma_{xz})$$

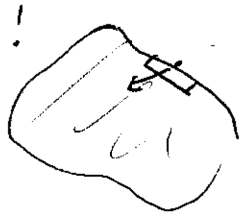
$$\Rightarrow \boxed{\sigma_{zx} = \sigma_{xz}} \checkmark$$

• (s. mentes) folyadék: $\underset{\sim}{\sigma_{ij} = -p\,\delta_{ij}}$ (álló foly.)

$$\underline{F}_V = \int_V d^3r\, \underline{f} + \int_{\partial V} \underline{\underline{\sigma}}\, d\underline{A} \Rightarrow F_i = \int_V f_i + \int_{\partial V} dA_k\, \sigma_{ik} = \int_V d^3r\, \big((\partial_k \sigma_{ik}) + f_i\big)$$

Egyensúly: $\boxed{f_i + \partial_k \sigma_{ik} = 0}$

határfelt!



$\underline{P} \cdot dA$ nyomja kívülről

$-\underline{\underline{\sigma}}\, d\underline{A} = -\underline{\underline{\sigma}}\, \underline{n}\, dA$ belülről

$$\Rightarrow \underline{P} \cdot dA - \underline{\underline{\sigma}}\, \underline{n}\, dA = 0$$

$$\Rightarrow \boxed{\underline{P} = \underline{\underline{\sigma}} \cdot \underline{n}}$$

pl.: – foly. nyomása:

$$\underset{\substack{\uparrow \\ -\hat{z}\cdot g}}{\rho \underline{g}} - \operatorname{grad} p = 0 \Rightarrow \operatorname{grad} p = -\rho \hat{z} g$$

$$\underset{\sim}{p = p_0 + \rho g (z_0 - z)}$$

– betonpillér ????

– homogén bolygó nyomása ?? $\Rightarrow$ p a Föld közepén ?

Mozgásegyenlet:

Newton: $P_i = \int_V d^3x\, \underset{\text{impulzussűrűség}}{\underset{\sim}{v_i \rho}}$

$$D_t P_i = \int_V d^3x\, (f_i + \partial_k \sigma_{ik})$$

$\Downarrow$

$$D_t P_i = \int_V d^3x\, \Big(\partial_t (v_i \rho) + \partial_k (v_k v_i \rho)\Big) = \int_V d^3x\, (f_i + \partial_k \sigma_{ik})$$

$$\Rightarrow \quad \boxed{\partial_t(v_i \varrho) + \partial_k(v_k v_i \varrho) = f_i + \partial_k \sigma_{ik}} \quad \sim \text{impulzusáramlás}$$

momentum impulzusáram. térf. erők belső erők

lehetőség: $v_i v_k \varrho - \sigma_{ik} \equiv P_{ik}$ "impulzusáram."

másik alak:

$$D_t \int d^3x\, \varrho A = \int d^3x \Big(\underbrace{d_t(\varrho A)}_{d_t \cancel{\varrho}\, A \,+\, \varrho\, d_t A} + \cancel{(\partial_i v_i)} \varrho A \Big) = \int d^3x\, \varrho\, d_t A$$

$A \to v_i$:

$$\boxed{\varrho\, d_t v_i = f_i + \partial_k \sigma_{ik}} \quad \Leftrightarrow \quad \underline{F} = m \cdot \underline{a}$$

$$\partial_t v_i = - v_k \partial_k v_i + \frac{1}{\varrho}\left(f_i + \partial_k \sigma_{ik}\right) \quad \rightarrow \text{ ha } \varrho(\underline{r},t),\ \underline{v}(\underline{r},t),\ f,\ \underline{\underline{\sigma}}(\underline{r},t) \text{ ismert } \Rightarrow \underline{v}(t+\Delta t, \underline{r})$$

Impulzusmom.?

$$\boxed{L_i = \int_V d^3x\, \epsilon_{ijk}\, x_j\, \varrho\, v_k}$$

nincs imp. mom. sűrűség!!

$$D_t L_i = \int_V d^3x\, \epsilon_{ijk}\, \varrho\, \underbrace{d_t(x_j v_k)}_{v_k \underbrace{(d_t x_j)}_{= v_j} + x_j d_t v_k} = \int d^3x\, \epsilon_{ijk}\, \varrho\, x_j\, d_t v_k$$

$\emptyset$

$$= \int d^3x\, \epsilon_{ijk}\, x_j\, f_k + \int d^3x\, \epsilon_{ijk}\, \underbrace{x_j \partial_e \sigma_{ke}}_{\partial_e(x_j \sigma_{ke}) - \delta_{je} \sigma_{ke}}$$

$$D_t L_i = \underbrace{\int d^3x\, \epsilon_{ijk}\, x_j f_k + \int \epsilon_{ijk}\, x_j\, (\sigma_{ke}\, dA_e)}_{M_i} + \int d^3x\, \epsilon_{ijk}\, \sigma_{jk}$$

Gauss

$\emptyset$ kell legyen $\Rightarrow \sigma_{jk} = \sigma_{kj}$!

impm.megm. ✓ ha $\sigma_{kj} = \sigma_{jk}$ azaz a belső erők forgatónyomatéka eltűnik!

Deformációk vs. feszültség:

$$\boxed{x_i' = x_i + s_i(x_i, t)}$$

elmozdulástér

$$x_i + \Delta x_i \to x_i'(\underline{x}+\Delta\underline{x}, t) = x_i + \Delta x_i + s_i + \Delta x_j \partial_j s_i + \dots$$

$$\Delta x_i' = (\delta_{ij} + \partial_j s_i)\Delta x_j$$

$$s_{ij} = \partial_j s_i = a_{ij} + \epsilon_{ij}$$

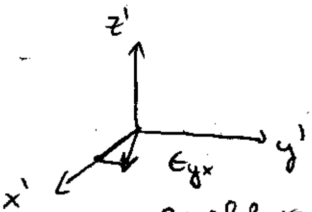
antisz. szimm.

forgatás!

$$a_{ij} = \frac{1}{2}(\partial_j s_i - \partial_i s_j)$$

$$\epsilon_{ij} = \dots + \dots$$

$$\Delta X \to \begin{pmatrix} \Delta x(1+\epsilon_{xx}) \\ \Delta x\, \epsilon_{yx} \\ \epsilon_{zx}\Delta x \end{pmatrix}$$



$\sim$ elforgatás szöge

hossza $\approx (1+\epsilon_{xx})\Delta x + \mathcal{O}(\epsilon^2)$

megnyúlás

térfogat vált.?

$$\int d^3x \to \int d^3x' = \int d^3x \left|\frac{\partial \underline{x}'}{\partial \underline{x}}\right|$$

pici

$$\det(1 + \partial_j s_i) \approx 1 + \mathrm{tr}\,\partial_j s_i = 1 + \mathrm{tr}\,\underline{\underline{\epsilon}}$$

$$\boxed{\frac{\Delta V'}{\Delta V} \cong 1 + \epsilon_{ii}}$$

folyadékok, szilárd testek: s_i nagy (cm, m)

$\underline{\underline{\text{de}}}$ $\partial_i s_j \sim \epsilon_{ij}$ pici $\frac{\Delta V'}{\Delta V} \approx 1$

Megjegyzés:

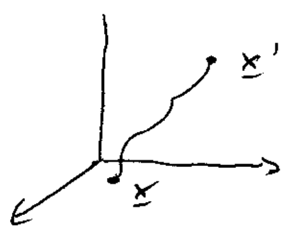
$$x_j' = x_i + s_i(\underline{x}, t)$$

$$\Rightarrow \boxed{v_i(\underline{x}', t) = \partial_t s_i(\underline{x}, t)}$$

nem $\underline{x}$!!

hasonlóan:

$$\boxed{S_{ij}(\underline{x}', t) = \frac{\partial}{\partial x_j} s_i(\underline{x}, t)}$$



Energiamérleg:

$$K = \int_V \frac{1}{2}\varrho v_i^2 \quad \Rightarrow \quad D_t K = \int_V \varrho v_i d_t v_i =$$

$$= \int_V v_i (f_i + \partial_j \sigma_{ij}) = \int_V v_i f_i - \int_V \sigma_{ij} \partial_j v_i$$

$$+ \int_V$$

Áll.: $\partial_j v_i = d_t s_{ij} = d_t \partial_j s_i$

biz.: $t: \quad x_i' = x_i + s_i(\underline{x}, t)$

$t+\Delta t: \quad x_i'' = x_i + s_i(\underline{x}, t+\Delta t) = x_i' + \Delta t\, v_i(\underline{x}', t)$

$$\frac{\partial x_i''}{\partial x_j} = \delta_{ij} + s_{ij}(\underline{x}'', t+\Delta t) = \underbrace{\frac{\partial x_i''}{\partial x_k'}}\ \underbrace{\frac{\partial x_k'}{\partial x_j}} = \delta_{ij} + s_{ij}(\underline{x}', t) + \Delta t\, \partial_k v_i(\underline{x}')$$

$$\left(\delta_{ik} + \Delta t \frac{\partial v_i}{\partial x_k'}\right)\left(\delta_{kj} + s_{kj}(\underline{x}', t)\right)$$

$$\Rightarrow \quad d_t s_{ij}(\underline{x}', t) = \partial_j v_i(\underline{x}', t) \qquad \square$$

ezt használva

$$D_t K = \int_V v_i f_i - \int_V \sigma_{ij} \underbrace{\partial_j v_i}_{d_t s_{ij} \to d_t \epsilon_{ij}} = \int_V \left(v_i f_i - \sigma_{ij} d_t \epsilon_{ij} + \partial_j(\sigma_{ji} v_i)\right)$$

$$\hookrightarrow \int_V \left(\partial_t\left(\frac{1}{2}\varrho v_i^2\right) + \partial_k\left(\frac{1}{2}\varrho v_i^2 \cdot v_k\right)\right)$$

$$\boxed{\partial_t \epsilon_{kin} + \partial_k (v_k \epsilon_{kin}) = v_i f_i - \sigma_{ij} d_t \epsilon_{ij} + \partial_j(\sigma_{ji} v_i)}$$

$\Rightarrow$ anyag rug. energiasűrűsége: $\boxed{d\phi = \sigma_{ij}\, d\epsilon_{ij}}$

(máshogy $\int (\partial_j \sigma_{ij})(\underline{x}')\, dx_i'\, d^3x'$)

általában

általában $dE = V \cdot \sigma_{ij}\, d\epsilon_{ij}$ $\rightarrow$ $\boxed{\sigma_{ij} = \frac{1}{V}\left(\frac{\partial E}{\partial \epsilon_{ij}}\right)_S}$

nyugalmas def. $\Rightarrow$ belső energia

véges T-n: $\sigma_{ij} = \left(\frac{\partial f}{\partial \epsilon_{ij}}\right)_T$ $\leftarrow$ szabadenergia sűrűség!

Deformálható anyag:

Hook-tv.: $\sigma_{ij} \propto \epsilon_{ij}$

$$\boxed{\sigma_{ij} = c_{ij,kl}\, \epsilon_{kl}}$$

$c_{ij,kl}$ nem mind független: $\boxed{f(\underline{\underline{\epsilon}}) = \frac{1}{2}\, \hat{c}_{ij,kl}\, \epsilon_{ij}\, \epsilon_{kl}}$

$\Rightarrow \sigma_{ij} = \hat{c}_{ij,kl}\, \epsilon_{kl}$

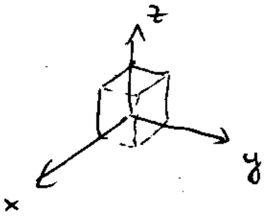
de
$$\hat{c}_{ij,kl} = \hat{c}_{ji,kl} = \hat{c}_{ij,lk} = \hat{c}_{kl,ij}$$

$\Rightarrow$ 21 db állandó

triklin kristály ...

-ös kristályban csak 3 db. független!

pl.



xz-re tükr.: $y \rightarrow -y$

$\Rightarrow \epsilon_{xy} \rightarrow -\epsilon_{xy}$

f viszont u.az

$\Rightarrow C_{xx,xy}\, \epsilon_{xx}\, \epsilon_{xy} \rightarrow -C_{xx,xy}\, \epsilon_{xx}\, \epsilon_{xy}$

$\Rightarrow C_{xx,xy} = 0$

független paraméterek:

$C_{xx,xx} \equiv A$; $C_{xx,yy} \equiv B$; $C_{xy,xy} \equiv C$

$$f = \frac{A}{2}\left(\epsilon_{xx}^2 + \epsilon_{yy}^2 + \epsilon_{zz}^2\right) + B\left(\epsilon_{xx}\epsilon_{yy} + \epsilon_{xx}\epsilon_{zz} + \epsilon_{yy}\epsilon_{zz}\right) + 2C\left(\epsilon_{xy}^2 + \epsilon_{xz}^2 + \epsilon_{yz}^2\right)$$

ekkor

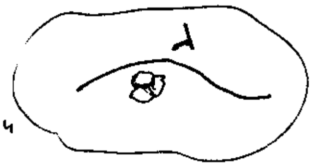
$$\sigma_{xx} = \frac{\partial f}{\partial \epsilon_{xx}} = A\,\epsilon_{xx} + B(\epsilon_{yy} + \epsilon_{zz})$$

$$\sigma_{xy} = \underset{\uparrow}{\frac{1}{2}} \frac{\partial f}{\partial \epsilon_{xy}} = 2C\,\epsilon_{xy}$$

$$df = \sigma_{xy}\,d\epsilon_{xy} + \sigma_{yx}\,d\epsilon_{yx} + \dots = 2\,\sigma_{xy}\,d\epsilon_{xy}$$

izotróp anyag: $\lambda \gg d_{\text{szemcse}}$

$\Rightarrow$ szemcsék orientációja véletlen



$\Rightarrow$ izotrópia!

forgatás: $\underline{\underline{\epsilon}} \to \underset{\substack{\uparrow \\ \text{forgatásmátrix}}}{R}\,\underline{\underline{\epsilon}}\,R^T \Rightarrow f(R\underline{\underline{\epsilon}}R^T) = f(\underline{\underline{\epsilon}})$

$f(\underline{\underline{\epsilon}})$ csak invariánsokon keresztül függhet $\underline{\underline{\epsilon}}$-től

$$\Rightarrow \boxed{f = \frac{\lambda}{2}(\mathrm{Tr}\,\underline{\underline{\epsilon}})^2 + \mu\,\mathrm{Tr}(\underline{\underline{\epsilon}}^2)} \qquad \text{Lamé-áll.}$$

ekkor: $\boxed{\underline{\underline{\sigma}} = 2\mu\,\underline{\underline{\epsilon}} + \lambda\,\mathrm{Tr}\,\underline{\underline{\epsilon}}}$ v. $\boxed{\sigma_{ij} = 2\mu\,\epsilon_{ij} + \lambda\,\delta_{ij}\,\epsilon_{kk}}$

tr.: $\Rightarrow$ (*) $\sigma_{kk} = (2\mu + 3\lambda)\,\epsilon_{kk}$

egyenletes összeny.: $\sigma_{kk} = -3p \Rightarrow \underset{\uparrow}{K} \cdot \frac{\delta V}{V} = -p$

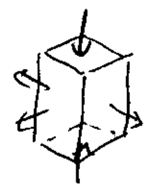
$K = \lambda + \frac{2}{3}\mu$ kompresszió-modulus

$$\left(\varkappa = \frac{1}{K}\right)$$

(*)-ból

- $\sigma_{ij} = K\,\delta_{ij}\,\epsilon_{kk} + 2\mu\,\big(\underbrace{\epsilon_{ij} - \tfrac{1}{3}\delta_{ij}\,\epsilon_{kk}}_{\text{spúrtalan}}\big)$

- $\boxed{\epsilon_{ij} = \frac{1}{2\mu}\,\sigma_{ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\,\delta_{ij}\,\sigma_{kk}}$ nyomás $\to$ válasz pl.: $\sigma_{ij} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & -p \end{pmatrix}$



$$\Rightarrow \quad \epsilon_{ij} = \begin{pmatrix} \frac{\lambda}{2\mu(2\mu+3\lambda)} & & \\ & \frac{\lambda}{2\mu(2\mu+3\lambda)} & \\ & & -\frac{\mu+\lambda}{\mu(2\mu+3\lambda)} \end{pmatrix} \cdot p$$

$$-\frac{1}{2\mu} + \frac{\lambda}{2\mu(2\mu+3\lambda)} = -\frac{\mu+\lambda}{\mu(2\mu+3\lambda)}$$

$$\epsilon_{zz} = \frac{\delta L}{L} = -\frac{\mu+\lambda}{\mu(2\mu+3\lambda)}\, p \quad \Rightarrow \quad E\,\frac{\delta L}{L} = -\frac{F}{A} \quad \Rightarrow \quad E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}$$

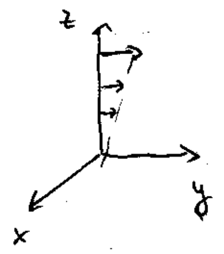
Young-mod.

$$\epsilon_{xx} = \frac{\delta L_x}{L_x} = \frac{\lambda}{2\mu(2\mu+3\lambda)}\, p \quad \Rightarrow \quad -\frac{\epsilon_{xx}}{\epsilon_{zz}} = \nu = \frac{\lambda}{2(\mu+\lambda)}$$

Poisson-szám

összeny. anyag ~ $\nu = \frac{1}{2}$, $\mu = 0$

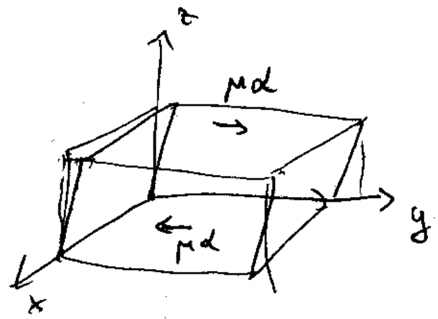
Nyírás:



$$\underline{s} = \begin{pmatrix} 0 \\ \alpha z \\ 0 \end{pmatrix} \Rightarrow \quad s_{ij} = \underset{z}{\partial_j}\, \underset{y}{s_i} = \begin{pmatrix} & & \\ & & \alpha \\ & & \end{pmatrix}$$

$$\Rightarrow \quad \epsilon_{ij} = \frac{1}{2}\begin{pmatrix} & & \\ & & \alpha \\ & \alpha & \end{pmatrix} \qquad \epsilon_{yz} = \epsilon_{zy} = \frac{\alpha}{2}$$

$$\Rightarrow \quad \sigma_{ij} = \begin{pmatrix} & & \\ & & \mu\alpha \\ & \mu\alpha & \end{pmatrix} \quad \Rightarrow$$



$$\underline{n} = \hat{z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \quad \underline{f} = \begin{pmatrix} 0 \\ \mu\alpha \\ 0 \end{pmatrix}$$

α szög $\Rightarrow \frac{F}{A} = \mu\alpha$

nyírási mod. $\rightarrow$ $\boxed{\mu = G}$

Hullámok kristályokban

$$\varrho\, d_t v_i = f_i + \partial_j \sigma_{ij}$$

Hook: $\sigma_{ij} = c_{ij,kl}\,\epsilon_{kl}$

$$v_i(\underline{x}',t) = \partial_t s_i(\underline{x},t) \qquad \epsilon_{kl}(\underline{x}',t) =$$

s pici $\Rightarrow$ • $v_i(\underline{x}') \approx v_i(\underline{x}) \approx \partial_t s_i(\underline{x},t)$

• $d_t v_i = \partial_t v_i + \underbrace{v_k \partial_k v_i}_{\mathcal{O}(s^2)} = \partial_t v_i$

• $\epsilon_{kl}(\underline{x}',t) \approx \epsilon_{kl}(\underline{x},t) + \mathcal{O}(s^2) = \frac{1}{2}(\partial_k s_l + \dots)$

$$\Rightarrow \varrho\, \partial_t^2 s_i = f_i + c_{ij,kl} \underbrace{\partial_j \epsilon_{kl}}_{\frac{1}{2}(\partial_j\partial_k s_l + \partial_j \partial_l s_k)} = f_i + c_{ij,kl}\,\partial_j \partial_k s_l$$

$$\boxed{\varrho\, \partial_t^2 s_i = f_i + c_{ij,kl}\,\partial_j \partial_k s_l}$$

izotróp anyag:

$$\sigma_{ij} = 2\mu\,\epsilon_{ij} + \lambda\,\delta_{ij}\,\mathrm{tr}\,\underline{\underline{\epsilon}}$$

$$\partial_j \sigma_{ij} = \mu\left(\partial_j \partial_i s_j + \partial_j^2 s_i\right) + \lambda\, \partial_i \partial_k s_k$$

$$= \left((\mu+\lambda)\,\mathrm{grad}(\mathrm{div}\,\underline{s}) + \mu\,\Delta \underline{s}\right)_i$$

$$\boxed{\varrho\,\partial_t^2 \underline{s} = (\mu+\lambda)\,\mathrm{grad}\,\mathrm{div}\,\underline{s} + \mu\,\Delta\underline{s}}$$

$\underline{s}(\underline{x},t) = \underline{e}\, e^{i(\underline{k}\cdot\underline{x} - \omega t)}$ megoldás v.

$$\underline{s}(\underline{x},t) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi}\, \underline{s}_{\underbrace{\omega,\underline{k}}_{Q}}\, e^{i\underline{k}\underline{x}}\, e^{-i\omega t}$$

$$\Rightarrow \quad \varrho\,\omega^2 \underline{s}_Q - (\mu+\lambda)\,\underline{k}(\underline{k}\,\underline{s}_Q) - \mu k^2 \underline{s}_Q = 0$$

$\underline{s}_Q \parallel \underline{k} \Rightarrow \varrho\,\omega^2 - (2\mu+\lambda)k^2 = 0 \qquad \underbrace{\frac{\omega}{k} = c_l}_{} = \sqrt{\frac{2\mu+\lambda}{\varrho}}$ ← longit módus

$\mathrm{rot}\,\underline{s} = \emptyset$

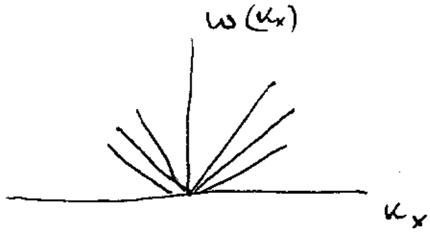
$\underline{s}_Q \perp \underline{k} \Rightarrow \underbrace{c_t = \sqrt{\frac{\mu}{\varrho}}}_{}$ ← torziós módus $\mathrm{div}\,\underline{s} = \emptyset$

(transverz) → foly → $\mu = 0$!

általános eset:

$$\underbrace{\left(\delta_{il}\,\rho\,(\omega^2/k^2) - c_{ijlk}\,\hat{k}_j\hat{k}_k\right)}\, s_l(Q) = 0$$

$$\det \ldots = 0 \quad \Rightarrow \quad \begin{aligned} \omega_1(\underline{k}) &= c_1(\hat{\underline{k}})\cdot k \\ \omega_2(\underline{k}) &= c_2(\hat{\underline{k}})\cdot k \\ \omega_3(\underline{k}) &= c_3(\hat{\underline{k}})\cdot k \end{aligned}$$

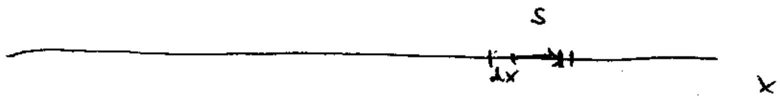


3 akusztikus módus

Folytonos közegek Lagrange-formalizmusa

~ térelmélet

egy D közeg: $s(x,t)$ elmozd. tér $\Rightarrow \rho\,\partial_t^2 s = c\,\partial_x^2 s$



$$\epsilon(x) \equiv \partial_x s(x)$$

kin energia: $\frac{1}{2}\,dx\,\rho\cdot(\partial_t s)^2 \Rightarrow K = \int dx\, \frac{\rho}{2}(\partial_t s)^2$

def. energia: $\frac{1}{2}\,c\,(\partial_x s)^2\,dx \Rightarrow U = \int dx\, \frac{\kappa}{2}(\partial_x s)^2$

$$\Rightarrow \boxed{L \equiv \int dx \underbrace{\left\{\frac{\rho}{2}(\partial_t s)^2 - \frac{\kappa}{2}(\partial_x s)^2\right\}}_{\mathcal{L}}} = K - U$$

$$\boxed{S \equiv \int_{t_1}^{t_2} dt \int dx \left[\frac{\rho}{2}(\partial_t s)^2 - \frac{\kappa}{2}(\partial_x s)^2\right]} = \int dt\, L$$

$$s(x,t) \rightarrow s(x,t) + \delta s(x,t) = s'(x,t) \quad \Rightarrow \partial_t \delta s = \delta \partial_t s$$

$$\delta S = \int_{t_1}^{t_2} dt \int dx \left(\rho\, \partial_t s\, \partial_t \delta s - \kappa\, \partial_x s\, \partial_x \delta s\right) =$$

$$= \int dx\, \rho\, \partial_t s\, \delta s(x,t)\Big|_{t_1}^{t_2} + \kappa \int dt\, \partial_x s \cdot \delta s\Big|_{-\infty}^{\infty} - \int dt\, dx \left(\rho\,\partial_t^2 s - \kappa\,\partial_x^2 s\right)\cdot\delta s$$

$$\delta s(x,t_1) \equiv \delta s(x,t_2) = \emptyset$$
$$\delta s(\infty,t) = \delta s(-\infty,t) \equiv \emptyset \quad \Rightarrow \quad \boxed{\delta S = 0}$$
$$\Leftrightarrow \quad \boxed{\varrho\, \partial_t^2 s = \mathcal{K}\, \partial_x^2 s}$$

Általános eset

$$\mathcal{L} = \mathcal{L}(\partial_t s_i,\ \partial_k s_i,\ s_i, \dots)$$

pl.: izotróp közeg: $\quad \mathrm{tr}\,\epsilon = \partial_k s_k$

$$\mathrm{tr}(\epsilon^2) = \frac{1}{4}(\partial_i s_j + \partial_j s_i)(\partial_i s_j + \partial_j s_i)$$
$$= \frac{1}{2}\left\{(\partial_i s_j)(\partial_i s_j) + \partial_i s_j \cdot \partial_j s_i\right\}$$

$$\Rightarrow \quad \mathcal{L} = \varrho\left(\frac{\partial s_i}{\partial t}\right)^2 - \frac{\lambda}{2}\,\partial_k s_k\, \partial_m s_m - \frac{\mu}{2}\left\{(\partial_i s_j)^2 + (\partial_i s_j)(\partial_j s_i)\right\}$$

$$S = \int L\, dt = \int dt\, d^3\underline{x}\ \mathcal{L}(\partial_t s_i,\ \partial_k s_i,\ s_i)$$

$$\delta S = \int dt\, d^3\underline{x} \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_t s_i)} \cdot \partial_t \delta s_i + \frac{\partial \mathcal{L}}{\partial(\partial_k s_i)}\, \partial_k \delta s_i + \frac{\partial \mathcal{L}}{\partial s_i}\, \delta s_i \right\}$$
$$= \int dt\, d^3\underline{x} \left\{ \frac{\partial \mathcal{L}}{\partial s_i} - \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t s_i)} - \partial_k \frac{\partial \mathcal{L}}{\partial(\partial_k s_i)} \right\} \delta s_i$$

„felületi tagok"
$$+ \int d^3\underline{x} \left[\frac{\partial \mathcal{L}}{\partial(\partial_t s_i)}\, \delta s_i \right]_{t_1}^{t_2} + \int dt \int dA_k\, \frac{\partial \mathcal{L}}{\partial(\partial_k s_i)}\, \delta s_i$$

$\to \emptyset$

$$\delta s_i(|\underline{x}| \to \infty, t) \equiv 0 \quad \forall t$$
$$\delta s_i(\underline{x}, t_1) = \delta s_i(\underline{x}, t_2) = \emptyset$$

$$\delta S = 0 \quad \Leftrightarrow \quad \boxed{\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t s_i)} + \partial_k \frac{\partial \mathcal{L}}{\partial(\partial_k s_i)} = \frac{\partial \mathcal{L}}{\partial s_i}}$$

Euler – Lagrange egy.

Folytonos def. közeg: $\qquad \mathcal{L} = \overset{\swarrow \text{cst}}{\frac{\varrho}{2}} (\partial_t s_i)^2 - \frac{1}{2} C_{ij,ke}\, \partial_i s_j\, \partial_k s_e$

$$\left.\begin{aligned} \frac{\partial \mathcal{L}}{\partial \partial_t s_i} &= \varrho\, \partial_t s_i \\ \frac{\partial \mathcal{L}}{\partial \partial_k s_i} &= C_{ik,em}\, \partial_e s_m \end{aligned}\right\} \Rightarrow \boxed{\varrho\, \partial_t^2 s_i - C_{ik,em}\, \partial_k \partial_e s_m = 0} \checkmark$$

Impulzusok ?

$$\sum_i \frac{m}{2} \dot{x}_i^2 = \sum_i \frac{m}{2} (\partial_t s)^2 = a \sum_i \frac{1}{2} \overset{\overset{\varrho}{\parallel}}{\frac{m}{a}} (\partial_t s)^2 = \int \frac{\varrho}{2} (\partial_t s)^2$$

$$x_i = x + s(x) \qquad x = i a$$

$$p_i = m \dot{x}_i = a \frac{m}{a} \partial_t s \quad \Rightarrow \quad \varrho\, \partial_t s = \frac{p_i}{a} \sim \text{imp. sűrűség}$$

$$\Rightarrow \quad \pi(\underline{x}, t) \equiv \frac{\partial \mathcal{L}}{\partial (\partial_t s)} \sim \pi_i \equiv \frac{\partial \mathcal{L}}{\partial (\partial_t s_i)}$$

$$\boxed{\mathcal{H} \equiv \pi_i \cdot \partial_t s_i - \mathcal{L}}$$

$$\mathcal{L} = \frac{\varrho}{2} (\partial_t s)^2 - \frac{\varkappa}{2} (\partial_x s)^2 \quad \Rightarrow \quad \mathcal{H} = \varrho (\partial_t s)^2 - \frac{\varrho}{2} (\partial_t s)^2 + \frac{\varkappa}{2} (\partial_x s)^2$$

$$\pi = \varrho\, \partial_t s \qquad \boxed{\mathcal{H} = \frac{1}{2\varrho} \pi^2 + \frac{\varkappa}{2} (\partial_x s)^2}$$

energia-sűrűség

Áll.: $\boxed{H \equiv \int d^3\underline{x}\; \mathcal{H} \qquad \text{megmarad}}$

biz.: $$\frac{dH}{dt} = \int d^3\underline{x} \left\{ \underbrace{\partial_t \left(\frac{\partial \mathcal{L}}{\partial \partial_t s} \partial_t s \right)} - \partial_t \mathcal{L} \right\}$$

$$= \partial_t^2 s \frac{\partial \mathcal{L}}{\partial \partial_t s} + \partial_t s\, \partial_t \left(\frac{\partial \mathcal{L}}{\partial \partial_t s} \right) = \partial_t^2 s \frac{\partial \mathcal{L}}{\partial (\partial_t s)} + \partial_t s \left(\frac{\partial \mathcal{L}}{\partial s} - \frac{\partial \mathcal{L}}{\partial (\partial_k s)} (\partial_k s) \right)$$

Euler – Lagr.

$$\partial_t \mathcal{L} = \underline{\frac{\partial \mathcal{L}}{\partial s} \partial_t s} + \frac{\partial \mathcal{L}}{\partial (\partial_k s)} \partial_t \partial_k s + \frac{\partial \mathcal{L}}{\partial (\partial_t s)} \partial_t^2 s$$

$$\Rightarrow \quad \frac{dH}{dt} = - \int d^3\underline{x} \; \partial_k \Big(\underbrace{\partial_t s_i \frac{\partial \mathcal{L}}{\partial(\partial_k s_i)}}_{} \Big)$$

j_E^k = energiaáram sűrűség kifelé!

rug. közeg $\Rightarrow$ $- v_i \cdot \sigma_{ki}$!

elegendően nagy felület

$$\Rightarrow \quad \frac{dH}{dt} = - \int dA_k \, j_E^k \Rightarrow \emptyset \qquad \square$$

Relativisztikus elméletek:

φ skalár $\quad dt\, d^3\underline{x}$ invariáns! ($|\det L| = 1$)

$$\Rightarrow \quad \boxed{L = \int d^3\underline{x} \; \mathcal{L}(\partial_\mu \varphi, \varphi)}$$

mozgásegy: $\quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) = \frac{\partial \mathcal{L}}{\partial \varphi}$

$$\Pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \quad \Rightarrow \quad \boxed{T^{\mu\nu} \equiv \Pi^\mu \partial^\nu \varphi - g^{\mu\nu} \mathcal{L}}$$

energiaimp tenzor

$T^{00} = \Pi^0 \partial^0 \varphi - \mathcal{L}$ energiasűrűség

állítás $\quad \boxed{\partial_\mu T^{\mu\nu} = \emptyset}$ $\leftarrow$ egyezik Fakó jel.-vel!

4 db. kont. egyenlet! $\quad T^{\mu 0}$ az energia

pl.:

- $\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi$

  $\Rightarrow \quad \partial^\mu \partial_\mu \varphi = 0 \quad$ ~ fény

- $\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \, \partial^\mu \varphi - \frac{1}{2} \frac{c^2 m^2}{\hbar^2} \varphi^2$

  $\Rightarrow \quad \partial_\mu \partial^\mu \varphi + \frac{m^2 c^2}{\hbar^2} \varphi = 0$

  m tömegű részecske!

  Klein-Gordon egy. Π-on!