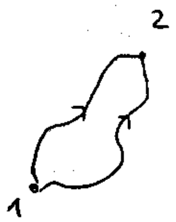
Három dimenziós mozgás, Kepler probléma

konzervatív erőterek:

$$\boxed{m\ddot{\underline{r}} = \underline{F}(\underline{r},t)}$$



$\underline{F}$ örvénymentes, ha $\displaystyle\int_{\underline{r}_1}^{\underline{r}_2} d\underline{r}\,\underline{F}(\underline{r},t)$

független az útvonaltól $\forall\, \underline{r}_1, \underline{r}_2$-re

$$V(\underline{r},t) = -\int_0^{\underline{r}} d\underline{r}'\cdot\underline{F}(\underline{r}',t) \quad \text{egyértelmű}, \quad \boxed{-\frac{\partial V}{\partial \underline{r}} = \underline{F}(\underline{r},t)}$$

$$\left(-\frac{\partial V}{\partial x^i} = F_i(\underline{r},t)\right)$$

$$\left[\; V(\underline{r}+\delta\underline{r},t) = V(\underline{r}) + (-1)\int_{\underline{r}}^{\underline{r}+\delta\underline{r}} d\underline{r}'\cdot\underline{F}(\underline{r}') \approx V(\underline{r}) - \sum_i \delta x^i\, F_i(\underline{r}) \;\right]$$

máshogy: $\underbrace{\oint d\underline{r}\,\underline{F}(\underline{r},t) = \emptyset}$ $\forall$ hurokra

$\qquad\qquad \shortparallel$

$\displaystyle\int d\underline{A}\cdot\mathrm{rot}\,\underline{F}$ $\forall$ felületre $\quad(\mathrm{rot}\,\underline{F}) = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \vdots \\ \end{pmatrix}$

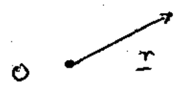
$$\Rightarrow \boxed{\mathrm{rot}\,\underline{F} = 0} \Leftrightarrow \boxed{\exists\; V(\underline{r},t) : \underline{F}(\underline{r},t) = -\frac{\partial V}{\partial \underline{r}}} \quad \text{potenciál}$$

$\underline{F}$ konzervatív ha $\exists$ $V(\underline{r},t) = V(\underline{r})$ időfüggetlen

ekkor
$$\boxed{E \equiv \underbrace{\frac{m}{2}\dot{\underline{r}}^2}_{\text{kinetikus energia}} + \underbrace{V(\underline{r})}_{\text{pot. energia}} = \text{cst}} \quad \text{energia megmarad}$$

biz. $\displaystyle\frac{d}{dt}\left(\frac{m}{2}\dot{\underline{r}}^2 + V(\underline{r})\right) = m\dot{\underline{r}}\cdot\ddot{\underline{r}} + \frac{\partial V}{\partial \underline{r}}\cdot\dot{\underline{r}} = \dot{\underline{r}}\left(m\ddot{\underline{r}} - \underline{F}(\underline{r})\right) = 0$ ✓

- impulzusmomentum:



adott ponthoz tartozó imp. mom:

$$\boxed{\underline{L} \equiv \underline{r}\times\underline{p}} \quad \left(\Longleftrightarrow L^i = \epsilon_{ijk}\, x^j p^k\right)$$

$$\boxed{\dot{\underline{L}}} = \underbrace{\dot{\underline{r}} \times \underline{p}}_{\emptyset} + \underline{r} \times \dot{\underline{p}} = \boxed{\underline{r} \times \underline{F}}$$ forgatónyomaték $\underline{N}$

mindkettő függ a von. rendszertől!!

Centrális erőtér

$$\boxed{\underline{F}(\underline{r}) = F(r)\,\hat{\underline{r}}} \Rightarrow V(r) = -\int_{r_0}^{r} F(r')\,dr'$$

gyakran $\to \infty$

$$-\left(\frac{\partial V}{\partial \underline{r}}\right)_i = -\frac{\partial V}{\partial x^i} = -\frac{\partial V}{\partial r} \cdot \frac{\partial r}{\partial x^i} = F(r) \cdot (\hat{\underline{r}})_i$$

$$r = \sqrt{x^2+y^2+z^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{1}{2}\frac{1}{r}2x = (\hat{\underline{r}})_x$$

tömegvonzás: $F(r) = -\gamma \frac{m_1 m_2}{r^2} \Rightarrow V(r) = -\frac{\gamma m_1 m_2}{r}$

Coulomb taszítás: $F(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} \Rightarrow \dots$

ellenőrizhető: $\text{rot}\,\underline{F} = 0$

$$(\text{rot}\,\underline{F})_z = \frac{\partial}{\partial x}\left(F(r)\frac{1}{r}y\right) - \frac{\partial}{\partial y}\left(F(r)\frac{1}{r}x\right) = \emptyset$$

$$\underline{N} = \underline{r} \times \underline{F} = 0 \Rightarrow \boxed{\underline{L} = \underline{r} \times m\underline{v} = cst}$$

$\Rightarrow$ síkbeli mozgás

választás: $\underline{L} \parallel \hat{\underline{z}}$.

hengerkoordináták $\begin{pmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ z \end{pmatrix} = \underline{r}$

$\underline{r}, \underline{v} \perp \underline{L} \Rightarrow z = \dot{z} = 0$

$$\Rightarrow \dot{\underline{r}} = \begin{pmatrix} \dot\rho\cos\varphi - \rho\sin\varphi\,\dot\varphi \\ \dot\rho\sin\varphi + \rho\cos\varphi\,\dot\varphi \\ 0 \end{pmatrix} \Rightarrow \frac{m\dot{\underline{r}}^2}{2} = \frac{m\dot\rho^2}{2} + \frac{m}{2}\rho^2\dot\varphi^2$$

$$\boxed{E = \frac{m\dot\rho^2}{2} + \frac{m}{2}\rho^2\dot\varphi^2 + V(r) = cst}$$

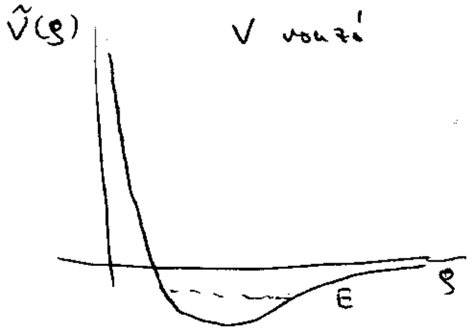
$$m\,\underline{r} \times \dot{\underline{r}} = \begin{pmatrix} 0 \\ 0 \\ m\rho\cos\varphi(\cancel{\dot\rho\sin\varphi} + \rho\cos\varphi\,\dot\varphi) - m\rho\sin\varphi(\cancel{\dot\rho\cos\varphi} - \rho\sin\varphi\,\dot\varphi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ m\rho^2\dot\varphi \end{pmatrix} = cst$$

$$\boxed{J = m\rho^2\dot\varphi =} \, cst \Leftrightarrow \text{Kepler II}$$

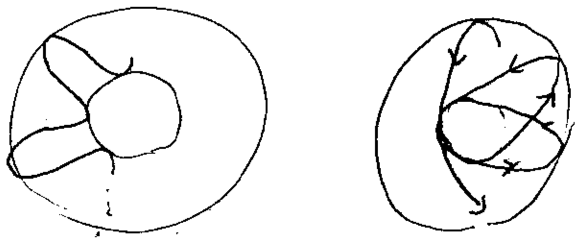
$dA = \frac{1}{2}\rho^2 d\varphi$

ρ

$$\Rightarrow \quad \boxed{E = \frac{m\dot{\varrho}^2}{2} + \underbrace{\frac{J^2}{2m\varrho^2} + V(\varrho)}_{\tilde{V}(\varrho)\ \text{effektív pot}} = \text{cst}}$$



$E < 0 \quad \Rightarrow \quad \varrho_{min} \leq \varrho \leq \varrho_{max}$



$$t - t_0 = \int_{\varrho_0}^{\varrho} \frac{d\varrho'}{\sqrt{\frac{2}{m}\left(E - \tilde{V}(\varrho')\right)}}$$

Kepler probléma:

pálya alakja: új változók: $t \to \varphi$, $\varrho \to u = \frac{1}{\varrho}$

$$\dot{\varrho} = \frac{d\varrho}{dt} = \frac{d\varrho}{d\varphi}\cdot\dot{\varphi} = \frac{J}{m\varrho^2}\frac{d\varrho}{d\varphi} = -\frac{J}{m}\frac{du}{d\varphi} \qquad V(r) = -\frac{\alpha}{r}$$

$$\Rightarrow \quad E = \frac{J^2}{2m}\left\{\left(\frac{du}{d\varphi}\right)^2 + u^2 - \frac{2m\alpha}{J^2}u\right\}$$

$$\tilde{u} \equiv u - \frac{m\alpha}{J^2} \quad \Rightarrow \quad \frac{2mE}{J^2} = \left(\frac{d\tilde{u}}{d\varphi}\right)^2 + \tilde{u}^2 - \frac{m^2\alpha^2}{J^4}$$

$$\left(\frac{d\tilde{u}}{d\varphi}\right)^2 + \tilde{u}^2 = \frac{2mE}{J^2}\left(1 + \frac{m\alpha^2}{2J^2E}\right) \equiv \mathcal{K}^2 = \frac{m^2\alpha^2}{J^4}\left(1 + \frac{2J^2E}{m\alpha^2}\right)$$

$$\frac{d\tilde{u}}{d\varphi} = \pm\left(\mathcal{K}^2 - \tilde{u}^2\right)^{1/2} \quad \Rightarrow \quad \frac{d^2\tilde{u}}{d\varphi^2} = \pm\frac{1}{2}\frac{1}{\sqrt{}}(-2\tilde{u})\frac{d\tilde{u}}{d\varphi} = -\tilde{u}$$

$$\boxed{\tilde{u} = \mathcal{K}\cos\varphi} \quad \Rightarrow \quad u = \frac{1}{\varrho(\varphi)} = \frac{1 + e\cos\varphi}{p} = \mathcal{K}\cos\varphi + \frac{m\alpha}{J^2}$$

$$\boxed{p = \frac{J^2}{m\alpha} \quad ; \quad e = \sqrt{1 + \frac{2J^2E}{m\alpha^2}}}$$

$$\boxed{\varrho(\varphi) = \frac{p}{1 + e\cos(\varphi - \varphi_0)}}$$

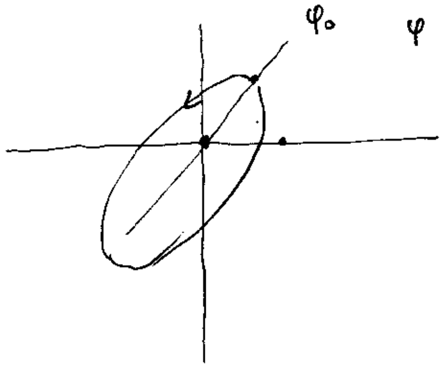
$p \sim$ pálya mérete

becslés : $\frac{\alpha}{r} \sim \frac{m r^2 \dot{\varphi}^2}{2} \sim \frac{J^2}{m r^2} \Rightarrow r \sim \frac{J^2}{m\alpha}$ ✓

e : excentricitás

$$e \begin{cases} <1 & \text{elliptikus} \quad (E<0) \\ 1 & \text{parabola} \quad (E=0) \\ >1 & \text{hiperbola} \quad (E>0) \end{cases}$$

$e<0$

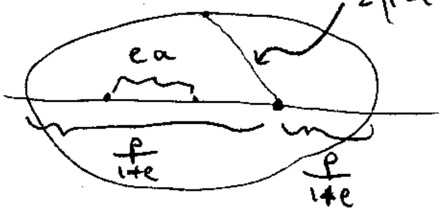


$\Leftrightarrow$ Kepler I. törvénye

$$\frac{p}{2}\left(\frac{1}{1-e}+\frac{1}{1+e}\right) = \frac{p}{1-e^2} = a$$

terület ?

$\varphi_0 = 0$



$\Rightarrow b = \sqrt{1-e^2} \cdot a$

$$A = \pi \cdot a \cdot b = \pi \cdot \frac{p^2}{(1-e^2)^{3/2}} \sim \frac{J^4}{(J^2|E|)^{3/2}} \frac{(m\alpha^2)^{3/2}}{m^2\alpha^2}$$

$$\Rightarrow \underset{\sim}{T} \sim \frac{mA}{J} \sim \underset{\sim\sim}{\frac{1}{|E|^{3/2}}} (m\alpha^2)^{1/2} \quad (J \text{ kiesik})$$

ugyanakkor $\quad \underset{\sim}{a} = \frac{p}{1-e^2} \sim \frac{J^2 m \alpha^2}{J^2 |E| m\alpha} \sim \underset{\sim}{\frac{1}{|E|}} \cdot \alpha$

$$\Rightarrow \boxed{\frac{T^2}{a^3} \sim \frac{m}{\alpha} \sim cst} \quad (\alpha \propto m)$$

Kepler III.

megj.: ez következik a mozgásegy. skálainv.-ból:

$$\ddot{\underline{r}} = -\frac{\gamma M_\odot}{r^3} \underline{r}$$

ha $\underline{r}(t)$ megoldás

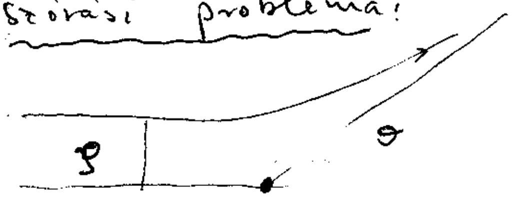
$\Rightarrow \tilde{\underline{r}}(\tilde{t}) = b_r \underline{r}(b_t t)$

is az, ha $\frac{b_r^3}{b_t^2} = 1$

$$\Rightarrow \frac{\tilde{a}^3}{\tilde{T}^2} = \frac{a^3}{T^2} = cst$$

Szórási probléma:

E



bejövő részecskék árama:

$$j = \frac{\#\ \text{részecskék}}{\text{idő} \cdot \text{felület}}$$

$$dN = d\Omega \cdot \frac{d\sigma(\vartheta)}{d\Omega} \cdot j \cdot \Delta t$$

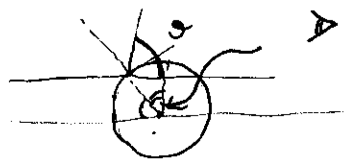
idő

differenciális hatásker

$$\frac{d\sigma}{d\Omega}(\vartheta)\, 2\pi \sin\vartheta\, |d\vartheta| = 2\pi \cdot \varrho \cdot |d\varrho|$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega}(\vartheta) = \frac{\varrho}{\sin\vartheta}\left|\frac{d\varrho}{d\vartheta}\right|}$$

Merev gömb:



$\alpha = \vartheta/2$

$$\Rightarrow \varrho = R \cdot \cos\frac{\vartheta}{2}$$

$$\left|\frac{d\varrho}{d\vartheta}\right| = \frac{R}{2}\sin\frac{\vartheta}{2}$$

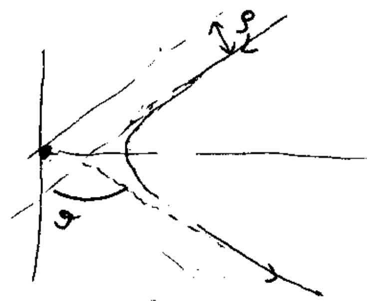
$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{R^2}{4}} \Rightarrow$$

$$\sigma_T = \int d\Omega \frac{d\sigma}{d\Omega} = R^2\pi$$ a gömb felülete (projektált)

Coulomb szórás

$$\alpha = -\frac{q_1 q_2}{4\pi\varepsilon_0} \Rightarrow p < 0 \ ; \ e > 1$$

$$r = \frac{|p|}{-1 + e\cos\varphi}$$



$$E = \frac{mv^2}{2}, \quad J = mv \cdot \varrho$$

$$\vartheta = \pi - 2\arccos\frac{1}{e} \Rightarrow \sin\frac{\vartheta}{2} = \frac{1}{e}$$

$$e = \sqrt{1 + \frac{2 J^2 E}{m \alpha^2}} = \left(1 + \frac{4 E^2 \varrho^2}{\alpha^2}\right)^{1/2}$$

$$\Rightarrow \quad \frac{4 E^2 \varrho^2}{\alpha^2} = \frac{1}{\sin^2 \frac{\vartheta}{2}} - 1 = \operatorname{ctg}^2 \frac{\vartheta}{2}$$

$$\Rightarrow \quad \varrho(\vartheta) = \frac{|\alpha|}{2E} \cdot \operatorname{ctg} \frac{\vartheta}{2} \quad ; \quad \left|\frac{d\varrho}{d\vartheta}\right| = \frac{|\alpha|}{4E} \cdot \frac{1}{\sin^2 \frac{\vartheta}{2}}$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{8E^2} \frac{1}{\sin^2 \frac{\vartheta}{2}} \operatorname{ctg} \frac{\vartheta}{2} \frac{1}{\sin \vartheta} = \frac{\alpha^2}{16 E^2} \frac{1}{\sin^4 \frac{\vartheta}{2}}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16 E^2} \frac{1}{\sin^4 \frac{\vartheta}{2}}}$$

- $\int$ -ja ∞ ...
- $E \to \infty \Rightarrow \frac{d\sigma}{d\Omega} \to \emptyset$ nagy energiájú részecskék gyengén szóródnak.