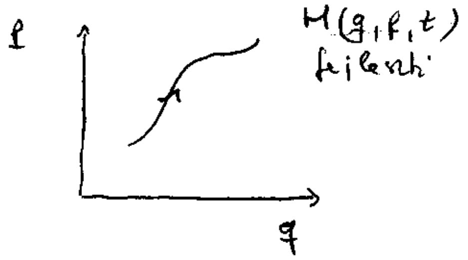
Kanonikus transzformációk

$\eta = (q, p) = (\{q_e\}, \{p_e\})$

$\{\eta_i\}$ $i = 1, \dots, 2f$ $\qquad e = 1, \dots, f$

η mozgásegyenlete Hamilton-féle: $\dfrac{d\eta_i}{dt} = J_{ik} \dfrac{\partial H(\eta, t)}{\partial \eta_k}$



új változók:

$Q_e \equiv Q_e(q, p, t)$

$P_e = P_e(q, p, t)$ $\Rightarrow \xi = (Q, P)$

$\Downarrow$

A $(q, p) \to (Q, P)$ trafó kanonikus, ha $\forall$ H-ra (!) $\exists$ $K(Q, P, t) = K(\xi, t)$

hogy $\boxed{\dot{\xi}_k = J_{kl} \dfrac{\partial K}{\partial \xi_l}}$

Állítás: $\eta \to \xi(\eta, t)$ kanonikus $\iff$ $\boxed{[\xi_i, \xi_j]_\eta = a J_{ij}}$ cst. $a \equiv 1$

azaz $[Q_e, P_m]_{q,p} = \delta_{em}$

és $[Q_e, Q_m]_{q,p} = [P_e, P_m]_{q,p} = 0$

nem bizonyítjuk, de néhány megjegyzést fűzünk hozzá:

- $[\xi_i, \xi_j]_\eta = \underbrace{\dfrac{\partial \xi_i}{\partial \eta_k}}_{M_{ik}} J_{kl} \underbrace{\dfrac{\partial \xi_j}{\partial \eta_l}}_{M^T_{lj}} = J_{ij} \Rightarrow M J M^T = J$

  szimplektikus mátrixok definíciója

- $[\xi_i, \xi_j]_\eta = J_{ij} \iff \forall$ f, g függvényre

  $[f, g]_\eta = [f, g]_\xi$

  ugyanis $\dfrac{\partial f}{\partial \eta_i} J_{ik} \dfrac{\partial g}{\partial \eta_k} = \dfrac{\partial f}{\partial \xi_l} \underbrace{\dfrac{\partial \xi_l}{\partial \eta_i} J_{ij} \dfrac{\partial \xi_k}{\partial \eta_j}}_{J_{lk}} \dfrac{\partial g}{\partial \xi_k}$ ✓

- inverz fv. tétel $\Rightarrow [\xi, \xi]_\eta = J \Rightarrow [\eta, \eta]_\xi = J$, azaz inverz is kanonikus

- fázistérf. (és Poincaré-inv.) invariáns kan. trafóra

• nagy szabadság!

pl. $Q_\ell = \alpha p_\ell \; ; \; \mathcal{P}_\ell = -q_\ell/\alpha$ kanonikus

$$[Q_\ell, \mathcal{P}_m] = [p_\ell, -q_m] = [q_m, p_\ell] = \delta_{\ell m}$$

• Ha $\xi_i(\eta, \cancel{t})$ (időfüggetlen), akkor $K(\xi) = H(\eta(\xi), t)$

$$\frac{d\xi_i}{dt} = \frac{\partial \xi_i}{\partial \eta_k}\frac{d\eta_k}{dt} = \frac{\partial \xi_i}{\partial \eta_k} J_{k\ell}\frac{\partial H}{\partial \eta_\ell} = \underbrace{\frac{\partial \xi_i}{\partial \eta_k} J_{k\ell}\frac{\partial \xi_m}{\partial \eta_\ell}}_{J_{im}}\frac{\partial H}{\partial \xi_m}$$

• KT-k sorozata is KT ~ csoportot alkotnak

egyszerű példák:

– $\dot{p} = pq \; ; \; \dot{q} = -pq$ nem Hamilton-féle rendszer

$$pq \overset{?}{=} -\frac{\partial H}{\partial q} \qquad -pq \overset{?}{=} \frac{\partial H}{\partial p} \quad \Rightarrow \quad \frac{\partial^2 H}{\partial p \partial q} - \frac{\partial^2 H}{\partial q \partial p} = -q + p \neq \emptyset \; \lightning$$

– $H \equiv p^2 q^2 \quad \Rightarrow \quad \dot{p} = -2qp^2 \qquad \dot{q} = 2p\cdot q^2$

– $Q \equiv q \; ; \; P \equiv \sqrt{p} - \sqrt{q}$ kanonikus?

nem! $[Q, P] = [q, \sqrt{p} - \sqrt{q}] = \frac{1}{2}\frac{1}{\sqrt{p}} \neq 1$

Kérdés: <u>hogyan találhatunk KT-t???</u>

① infinitezimális KT-k

② alkotófv-ek módszere

<u>Infinitezimális KT-k:</u>

$$\xi = \eta + \delta\eta = \eta + \delta\lambda J \frac{\partial G(\eta, t)}{\partial \eta}$$

$$\boxed{\xi = \eta + \delta\eta \; ; \; \delta\eta = \delta\lambda[\eta, G]} \quad \Rightarrow \quad \xi_i = \eta_i + \delta\lambda J_{ik}\frac{\partial G}{\partial \eta_k}$$

ekkor $M_{ik} = \frac{\partial \xi_i}{\partial \eta_k} = \delta_{ik} + \delta\lambda J_{im}\frac{\partial^2 G}{\partial \eta_m \partial \eta_k}$ $\equiv G^{(2)}_{m,k} \equiv \Gamma$

$$\Rightarrow [\xi_i, \xi_\ell]_\eta = (M J M^T)_{i\ell} = \Big((1 + \delta\lambda J\Gamma)J\underbrace{(1 + \delta\lambda J\Gamma)^T}_{1 - \delta\lambda \Gamma J}\Big)_{i\ell}$$

$$= \left(1 + \delta\lambda J\Gamma J - \delta\lambda J\Gamma J + \ldots\right)_{i\ell} = \delta_{i\ell} + \mathcal{O}(\delta\lambda)^2$$

$$\frac{d\xi_i}{dt} = \underbrace{\frac{\partial \xi_i}{\partial \eta_k} J_{ke} \frac{\partial H}{\partial \eta_e}}_{J_{ie}\frac{\partial H}{\partial \xi_e} + \mathcal{O}(\delta\lambda^2)} + \delta\lambda \, J_{ik} \frac{\partial}{\partial t} \underbrace{\frac{\partial G}{\partial \eta_k}}_{\frac{\partial G}{\partial \xi_k} + \mathcal{O}(\delta\lambda)} = J_{ik} \frac{\partial}{\partial \xi_k}\left(H + \delta\lambda \frac{\partial G}{\partial t}\right)$$

$$\Rightarrow \quad K(\xi, t) = H(\eta(\xi,t), t) + \delta\lambda \underbrace{\frac{\partial G}{\partial t}}_{} (\eta(\xi,t), t)$$

vezető rendben $\eta \cong \xi$

$$\boxed{K = H + \delta\lambda \frac{\partial G}{\partial t}}$$

$$\eta(\xi, t) = \xi - \delta\lambda \, J \frac{\partial G}{\partial \xi}$$

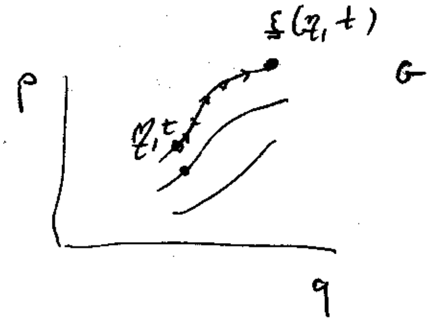
$$K(\xi, t) = H\left(\xi - \delta\lambda J \frac{\partial G}{\partial \xi}, t\right) + \delta\lambda \frac{\partial G(\xi, t)}{\partial t}$$

- következmények: időbeli eltolás $\eta(0) \to \eta(t)$ KT !

  $G \equiv H \Rightarrow \delta\eta = \delta t \, J \frac{\partial H}{\partial \eta}$ - ✓

  - forgatás kan. trafó (L_z)
  - eltolás KT (P_x)

  sok apró KT → egy nagy



- Alkotófüggvények módszere!

  módosított H. elv:

  $(Q(t), P(t))$ kiel. mozgásegy. $\Rightarrow \delta \int (P \cdot \dot{Q} - K(P, Q))\,dt = 0$

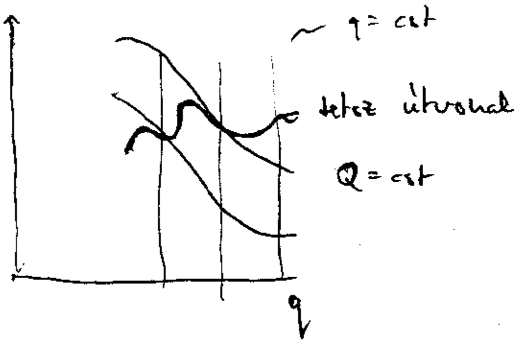
  hasonlóan

  $(q(t), p(t))$ kiel. Ham. egy. $\Rightarrow \delta \int (p \cdot \dot{q} - H(p, q, t))\,dt = 0$

  $\Big\}\ \forall\ \delta p, \delta q$ -ra

  egyidejűleg kielégülnek, ha $\boxed{P \cdot \dot{Q} - K(P, Q, t) = p\dot{q} - H - \frac{dF}{dt}}$

  - közös $p(t)$ és $q(t)$ görbén
  - $P(p(t), q(t), t)$, $Q = Q(p(t), q(t), t)$



t rögz.

$$\Rightarrow \quad P_e = P_e(q, Q, t)$$
$$p_e = p_e(q, Q, t)$$
$$F(q, Q, t)$$

akkor

$$\sum_e P_e(q,Q,t)\,\dot{Q}_e - K(q,Q,t) = \sum_e p_e(q,Q,t)\cdot\dot{q}_e - H(q,Q,t)$$
$$- \frac{\partial F}{\partial q_e}\dot{q}_e - \frac{\partial F}{\partial Q_e}\dot{Q}_e - \frac{\partial F}{\partial t} \qquad \forall\ q, Q, \dot{Q}, \dot{q}, t\text{-re}$$

$$\Rightarrow \quad \boxed{P_e(q,Q,t) = -\frac{\partial F}{\partial Q_e}}\ ; \quad \boxed{p_e = \frac{\partial F}{\partial q_e}}\ ; \quad \boxed{K = \frac{\partial F}{\partial t} + H}$$

első fajú / fajta

Pl.:

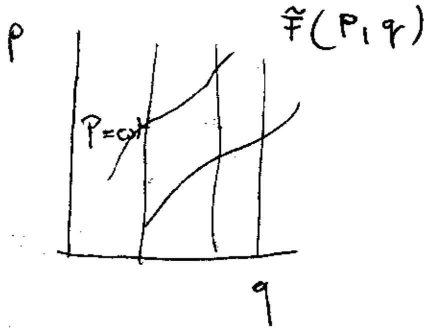
- $F \equiv q\cdot Q \quad \Rightarrow \quad P = -q\ ;\ p = Q$
- $F = q^2 Q^2 \quad \Rightarrow \quad P = -2Qq^2\ ;\ p = 2q\cdot Q^2$

$$\Downarrow$$
$$Q = \pm\sqrt{\frac{p}{2q}}$$

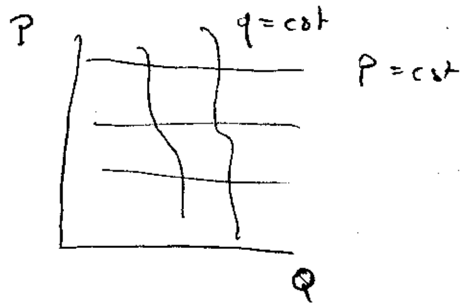
de használhatunk más koordinátát is

$$F \equiv \tilde{F} - P\cdot Q$$

$$\Rightarrow \quad -\dot{P}Q - K = p\dot{q} - H - \frac{d\tilde{F}}{dt} \quad \leftarrow \quad \tilde{F}(P, q)$$



$\Leftrightarrow$



$$\frac{d\tilde{F}}{dt} = \frac{\partial\tilde{F}}{\partial P}\dot{P} + \frac{\partial\tilde{F}}{\partial q}\dot{q} + \frac{\partial\tilde{F}}{\partial t}$$

$$\Rightarrow \quad \boxed{Q = \frac{\partial\tilde{F}}{\partial P}\ ; \quad p = \frac{\partial\tilde{F}}{\partial q}\ ; \quad K = H + \frac{\partial\tilde{F}}{\partial t}}$$

Példák:

- $\tilde{F} = q\mathcal{P} \quad \Rightarrow \quad Q = \frac{\partial \tilde{F}}{\partial \mathcal{P}} = q \; ; \quad p = \frac{\partial \tilde{F}}{\partial q} = \mathcal{P}$

  identitás ...

  $\Rightarrow$ infinit. trafó: $\quad \tilde{F} = q\mathcal{P} + \epsilon G(q, \mathcal{P}, t)$

$$Q = q + \epsilon \frac{\partial G}{\partial \mathcal{P}} \approx q + \epsilon \frac{\partial G}{\partial p}$$

$$p = \frac{\partial \tilde{F}}{\partial q} = \mathcal{P} + \epsilon \frac{\partial G}{\partial q} \quad \Rightarrow \quad \mathcal{P} = p - \epsilon \frac{\partial G}{\partial q}$$

$$\boxed{K = H + \epsilon \frac{\partial G}{\partial t}} \quad \Rightarrow \quad \boxed{\delta z = \epsilon J \frac{\partial G}{\partial z}}$$

- ponttranszformáció: $\quad \tilde{F} = f(q)\mathcal{P}$

$$\Rightarrow \quad Q = f(q) \; ; \quad p = \frac{\partial f}{\partial q} \mathcal{P} \qquad !$$

Harm. oszcillátor:

$$H = \frac{1}{2m}p^2 + \frac{m\omega^2}{2} q^2 = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right)$$

$$H = \text{cst} \quad \Rightarrow \quad p = A \cdot \cos Q \; ; \quad q = \frac{A}{m\omega} \sin Q$$

$$\Rightarrow \quad p = m\omega q \operatorname{ctg} Q$$

$$\exists ? \; F(q, Q) \qquad p = +\frac{\partial F}{\partial q} \quad \Rightarrow \quad F = \frac{m\omega}{2} q^2 \operatorname{ctg} Q$$

$$\Rightarrow \quad \mathcal{P} = -\frac{\partial F}{\partial Q} = \frac{m\omega}{2} q^2 \frac{1}{\sin^2 Q} \quad \Rightarrow \quad q = \sqrt{\frac{2\mathcal{P}}{m\omega}} \sin Q$$

$$\Rightarrow \quad \boxed{H(\mathcal{P}, Q) = \omega \mathcal{P}} \qquad \boxed{\dot{Q} = \omega}$$

$\mathcal{P} \equiv I$ hatásvált. $\qquad Q \equiv \phi$ szögváltozó