

**Math 241 - Calculus III**  
**Spring 2012, section CL1**  
**§ 16.8. Stokes' theorem**

In these notes, we illustrate Stokes' theorem by a few examples, and highlight the fact that many different surfaces can bound a given curve.

## 1 Statement of Stokes' theorem

Let  $S$  be a surface in  $\mathbb{R}^3$  and let  $\partial S$  be the boundary (curve) of  $S$ , oriented according to the usual convention. That is, "if we move along  $\partial S$  and fall to our left, we hit the side of the surface where the normal vectors are sticking out". Let  $\vec{F}$  be a vector field that is defined (and smooth) in a neighborhood of  $S$ . Then the following equality holds:

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} \, dS.$$

The theorem can be useful in either direction: sometimes the line integral is easier than the surface integral, sometimes the other way around.

## 2 Examples

Consider the vector field  $\vec{F} = (y, xz, 1)$  whose curl is

$$\begin{aligned} \text{curl } \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & 1 \end{vmatrix} \\ &= \vec{i}(0 - x) - \vec{j}(0 - 0) + \vec{k}(z - 1) \\ &= (-x, 0, z - 1). \end{aligned}$$

Consider the curve  $C$  which is the unit circle in the  $xy$ -plane, defined by  $x^2 + y^2 = 1$ ,  $z = 0$ , oriented counterclockwise when viewed from above. Let us compute the line integral of  $\vec{F}$  along  $C$ .

First, parametrize  $C$  by the usual "longitude" angle  $\theta$ :

$$\begin{aligned} \vec{r}(\theta) &= (\cos \theta, \sin \theta, 0) \\ 0 &\leq \theta \leq 2\pi. \end{aligned}$$

Then we have

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (y, xz, 1) \cdot \vec{r}'(t) dt \\ &= \int_0^{2\pi} (\sin \theta, 0, 1) \cdot (-\sin \theta, \cos \theta, 0) dt \\ &= \int_0^{2\pi} -\sin^2 \theta dt \\ &= \boxed{-\pi}. \end{aligned}$$

Stokes' theorem claims that if we “cap off” the curve  $C$  by any surface  $S$  (with appropriate orientation) then the line integral can be computed as

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} dS.$$

Now let's have fun! More precisely, let us verify the claim for various choices of surface  $S$ .

## 2.1 Disk

Take  $S$  to be the unit disk in the  $xy$ -plane, defined by  $x^2 + y^2 \leq 1$ ,  $z = 0$ . According to the orientation convention, the normal  $\vec{n}$  to  $S$  should be oriented upward, so that in fact  $\vec{n} = (0, 0, 1)$ .

$$\begin{aligned} \iint_S \text{curl } \vec{F} \cdot \vec{n} dS &= \iint_S (-x, 0, z - 1) \cdot (0, 0, 1) dS \\ &= \iint_S (z - 1) dS \\ &= \iint_S (-1) dS \\ &= -\text{Area}(S) \\ &= \boxed{-\pi}. \end{aligned}$$

## 2.2 Hemisphere

Take  $S$  to be the unit **upper hemisphere**, defined by  $x^2 + y^2 + z^2 = 1$ ,  $z \geq 0$ . According to the orientation convention, the normal  $\vec{n}$  to  $S$  should be oriented upward, pointing away from the origin. That means  $\vec{n} = \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} = (x, y, z)$ . Let us parametrize  $S$  in spherical coordinates,

with colatitude  $0 \leq \phi \leq \frac{\pi}{2}$  and longitude  $0 \leq \theta \leq 2\pi$ .

$$\begin{aligned}
\iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, dS &= \iint_S (-x, 0, z-1) \cdot (x, y, z) \, dS \\
&= \iint_S -x^2 + z^2 - z \, dS \\
&= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} (-(\sin \phi \cos \theta)^2 + (\cos \phi)^2 - \cos \phi) \sin \phi \, d\theta \, d\phi \\
&= \int_0^{\frac{\pi}{2}} (-\pi \sin^2 \phi + 2\pi \cos^2 \phi - 2\pi \cos \phi) \sin \phi \, d\phi \\
&= \pi \int_0^{\frac{\pi}{2}} (\cos^2 \phi - 1 + 2 \cos^2 \phi - 2 \cos \phi) \sin \phi \, d\phi \\
&= \pi \int_0^{\frac{\pi}{2}} (3 \cos^2 \phi - 2 \cos \phi - 1) \sin \phi \, d\phi \quad u = \cos \phi, \, du = -\sin \phi \, d\phi \\
&= \pi \int_1^0 (3u^2 - 2u - 1)(-du) \\
&= \pi [u^3 - u^2 - u]_0^1 \\
&= \pi(1 - 1 - 1) \\
&= \boxed{-\pi}.
\end{aligned}$$

We could also take  $S$  to be the unit **lower hemisphere**, defined by  $x^2 + y^2 + z^2 = 1$ ,  $z \leq 0$ . According to the orientation convention, the normal  $\vec{n}$  to  $S$  should be oriented upward, pointing towards the origin. That means  $\vec{n} = \frac{-(x,y,z)}{\sqrt{x^2+y^2+z^2}} = -(x, y, z)$ . Again, we parametrize  $S$  in spherical coordinates, with colatitude  $\frac{\pi}{2} \leq \phi \leq \pi$  and longitude  $0 \leq \theta \leq 2\pi$ . A very similar calculation yields:

$$\begin{aligned}
\iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, dS &= \iint_S (-x, 0, z-1) \cdot (-(x, y, z)) \, dS \\
&= - \iint_S -x^2 + z^2 - z \, dS \\
&= -\pi \int_{\frac{\pi}{2}}^{\pi} (3 \cos^2 \phi - 2 \cos \phi - 1) \sin \phi \, d\phi \\
&= -\pi \int_0^{-1} (3u^2 - 2u - 1)(-du) \\
&= \pi [u^3 - u^2 - u]_0^{-1} \\
&= \pi((-1) - 1 - (-1)) \\
&= \boxed{-\pi}.
\end{aligned}$$

## 2.3 Paraboloid

Take  $S$  to be the part of the paraboloid defined by  $z = 1 - x^2 - y^2$ ,  $x^2 + y^2 \leq 1$ . According to the orientation convention, the normal  $\vec{n}$  to  $S$  should be oriented upward, pointing away from the  $z$ -axis.

Let us parametrize  $S$  using  $x$  and  $y$  as parameters, with domain of parametrization  $D$  the unit disk  $x^2 + y^2 \leq 1$ :

$$\vec{r}(x, y) = (x, y, 1 - x^2 - y^2).$$

The tangent vectors are

$$\vec{r}_x = (1, 0, -2x)$$

$$\vec{r}_y = (0, 1, -2y)$$

so that a normal vector is given by the cross product

$$\vec{r}_x \times \vec{r}_y = (2x, 2y, 1).$$

Check the orientation: This normal vector points up and away from the  $z$ -axis. Ok! No need to flip it.

$$\begin{aligned} \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, dS &= \iint_D \operatorname{curl} \vec{F} \cdot (\vec{r}_x \times \vec{r}_y) \, dx dy \\ &= \iint_D (-x, 0, z - 1) \cdot (2x, 2y, 1) \, dx dy \\ &= \iint_D -2x^2 + z - 1 \, dx dy \\ &= \iint_D -2x^2 + (1 - x^2 - y^2) - 1 \, dx dy \\ &= \iint_D -3x^2 - y^2 \, dx dy \\ &= - \int_0^{2\pi} \int_0^1 (3r^2 \cos^2 \theta + r^2 \sin^2 \theta) \, r dr d\theta \\ &= - \int_0^{2\pi} \int_0^1 (3r^3 \cos^2 \theta + r^3 \sin^2 \theta) \, dr d\theta \\ &= - \int_0^{2\pi} \frac{3}{4} \cos^2 \theta + \frac{1}{4} \sin^2 \theta \, d\theta \\ &= - \left( \frac{3}{4}(\pi) + \frac{1}{4}(\pi) \right) \\ &= \boxed{-\pi}. \end{aligned}$$

## 2.4 Cone

Take  $S$  to be the part of the cone defined by  $z = 1 - \sqrt{x^2 + y^2}$ ,  $x^2 + y^2 \leq 1$ . According to the orientation convention, the normal  $\vec{n}$  to  $S$  should be oriented upward, pointing away from the

$z$ -axis.

Let us parametrize  $S$  in cylindrical coordinates with  $0 \leq r \leq 1$  and  $0 \leq \theta \leq 2\pi$ :

$$\vec{r}(r, \theta) = (r \cos \theta, r \sin \theta, 1 - r).$$

The tangent vectors are

$$\begin{aligned}\vec{r}_r &= (\cos \theta, \sin \theta, -1) \\ \vec{r}_\theta &= (-r \sin \theta, r \cos \theta, 0)\end{aligned}$$

so that a normal vector is given by the cross product

$$\begin{aligned}\vec{r}_r \times \vec{r}_\theta &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos \theta & \sin \theta & -1 \\ -r \sin \theta & r \cos \theta & 0 \end{vmatrix} \\ &= \vec{i}(0 - (-r \cos \theta)) - \vec{j}(0 - r \sin \theta) + \vec{k}(r \cos^2 \theta - (-r \sin^2 \theta)) \\ &= (r \cos \theta, r \sin \theta, r).\end{aligned}$$

Check the orientation: This normal vector points up and away from the  $z$ -axis. Ok! No need to flip it.

$$\iint_S \text{curl } \vec{F} \cdot \vec{n} \, dS = \iint_D \text{curl } \vec{F} \cdot (\vec{r}_r \times \vec{r}_\theta) \, dr d\theta$$

Notice! The vector  $\vec{r}_r \times \vec{r}_\theta = (r \cos \theta, r \sin \theta, r)$  already encodes the distortion of area  $dS = |\vec{r}_r \times \vec{r}_\theta| \, dr d\theta = \sqrt{2} r \, dr d\theta$ . Do not throw in an extra  $r$ .

$$\begin{aligned}&= \iint_D (-x, 0, z - 1) \cdot (r \cos \theta, r \sin \theta, r) \, dr d\theta \\ &= \iint_D (-r^2 \cos^2 \theta + (-r)r) \, dr d\theta \\ &= - \int_0^{2\pi} \int_0^1 r^2 (\cos^2 \theta + 1) \, dr d\theta \\ &= -\frac{1}{3} \int_0^{2\pi} (\cos^2 \theta + 1) \, d\theta \\ &= -\frac{1}{3}(\pi + 2\pi) \\ &= \boxed{-\pi}.\end{aligned}$$

## 2.5 Tin can

Take  $S$  to be the “tin can with a top but without the bottom”, of height, say, 5. In other words,  $S$  consists of the part of the cylinder  $x^2 + y^2 = 1$ ,  $0 \leq z \leq 5$  (the side of the can), along with the disk  $x^2 + y^2 \leq 1$ ,  $z = 5$  (the top of the can). Call them respectively  $S_1$  and  $S_2$ .

According to the orientation convention, the normal  $\vec{n}$  to  $S_1$  should point away from the  $z$ -axis. That means  $\vec{n} = (x, y, 0)$ . Let us parametrize  $S_1$  in cylindrical coordinates with  $0 \leq z \leq 5$  and  $0 \leq \theta \leq 2\pi$ :

$$\vec{r}(z, \theta) = (\cos \theta, \sin \theta, z).$$

$$\begin{aligned} \iint_{S_1} \operatorname{curl} \vec{F} \cdot \vec{n} \, dS &= \iint_{S_1} (-x, 0, z - 1) \cdot (x, y, 0) \, dS \\ &= \iint_{S_1} -x^2 \, dS \\ &= - \int_0^{2\pi} \int_0^5 (\cos^2 \theta)(1) \, dz \, d\theta \\ &= -5 \int_0^{2\pi} (\cos^2 \theta) \, d\theta \\ &= -5\pi. \end{aligned}$$

The normal  $\vec{n}$  to  $S_2$  should point up. That means  $\vec{n} = (0, 0, 1)$ .

$$\begin{aligned} \iint_{S_2} \operatorname{curl} \vec{F} \cdot \vec{n} \, dS &= \iint_{S_2} (-x, 0, z - 1) \cdot (0, 0, 1) \, dS \\ &= \iint_{S_2} (z - 1) \, dS \\ &= \iint_{S_2} (4) \, dS \\ &= 4 \operatorname{Area}(S_2) \\ &= 4\pi. \end{aligned}$$

Combining the two parts, we obtain

$$\begin{aligned} \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, dS &= \iint_{S_1} \operatorname{curl} \vec{F} \cdot \vec{n} \, dS + \iint_{S_2} \operatorname{curl} \vec{F} \cdot \vec{n} \, dS \\ &= -5\pi + 4\pi \\ &= \boxed{-\pi}. \end{aligned}$$

## 3 Closed surfaces

### 3.1 Closed curves

Recall the following from chapter 13.

**Definition 3.1.** A **closed curve** is a curve that ends where it started.

In other words, a closed curve  $C$  has no endpoints floating around; it forms a loop. Another way to say this is that its boundary is empty:  $\partial C = \emptyset$ . In general, the boundary of a curve

is its ending point minus its starting point (the signs account for orientation). If the curve  $C$  goes from  $A$  to  $B$ , then we can write

$$\partial C = B - A.$$

A consequence of the fundamental theorem of line integrals is that integrating a **conservative** vector field along a **closed** curve  $C$  automatically yields zero:

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A) = 0. \quad (1)$$

In fact, we learned that property (1) characterizes conservative vector fields: A vector field is conservative **if and only if** its integral along any loop is zero.

Property (1) is not as mysterious as it seems. The key is that conservative vector fields are very special. Most vector fields are not conservative, i.e. are not the gradient of any function.

## 3.2 In 2 dimensions

What is the analogous notion for 2-dimensional objects, namely surfaces?

**Definition 3.2.** A **closed surface** is a surface that has no boundary.

In other words, a closed surface  $S$  has no “edge” floating around. Another way to say this is that its boundary is empty:  $\partial S = \emptyset$ . In general, the boundary of a surface will be a curve, or possibly several curves.

*Example 3.3.* Let  $S$  is the upper hemisphere of radius  $R$ , defined by  $x^2 + y^2 + z^2 = R^2$ ,  $z \geq 0$ . Its boundary  $\partial S$  is the circle of radius  $R$  in the  $xy$ -plane, defined by  $x^2 + y^2 = R^2$ ,  $z = 0$ .

*Example 3.4.* Let  $S$  is the sphere of radius  $R$ , defined by  $x^2 + y^2 + z^2 = R^2$ . Its boundary  $\partial S$  is empty. That is, the sphere is a **closed** surface.

*Example 3.5.* Let  $S$  is the part of the cylinder of radius  $R$  around the  $z$ -axis, of height  $H$ , defined by  $x^2 + y^2 = R^2$ ,  $0 \leq z \leq H$ . Its boundary  $\partial S$  consists of two circles of radius  $R$ :  $C_1$  defined by  $x^2 + y^2 = R^2$ ,  $z = 0$ , and  $C_2$  defined by  $x^2 + y^2 = R^2$ ,  $z = H$ .

A consequence of Stokes’ theorem is that integrating a vector field which is a **curl** along a **closed** surface  $S$  automatically yields zero:

$$\begin{aligned} \iint_S \text{curl } \vec{F} \cdot \vec{n} \, dS &= \int_{\partial S} \vec{F} \cdot d\vec{r} \\ &= \int_{\emptyset} \vec{F} \cdot d\vec{r} \\ &= 0. \end{aligned} \quad (2)$$

*Remark 3.6.* In case the idea of integrating over an empty set feels uncomfortable – though it shouldn’t – here is another way of thinking about the statement. If  $S$  is a closed surface, cut it into two parts  $S_1$  and  $S_2$  along some curve  $C$ . For example, we can cut the sphere  $S$  into the upper hemisphere  $S_1$  and lower hemisphere  $S_2$  along the equator  $C$ . Applying Stokes’s theorem

to each part yields

$$\begin{aligned} \iint_S \operatorname{curl} \vec{F} \cdot \vec{n} \, dS &= \iint_{S_1} \operatorname{curl} \vec{F} \cdot \vec{n} \, dS + \iint_{S_2} \operatorname{curl} \vec{F} \cdot \vec{n} \, dS \\ &= \int_C \vec{F} \cdot d\vec{r} - \int_C \vec{F} \cdot d\vec{r} \\ &= 0 \end{aligned}$$

where the opposite signs come from the orientation convention.

In fact, property (2) characterizes curls: A vector field is the curl of some vector field **if and only if** its integral along any closed surface is zero.

Property (2) is not as mysterious as it seems. The key is that curls are very special. Most vector fields are not the curl of a vector field.

## 4 Which vector fields are curls?

We have seen that vector fields of the form  $\operatorname{curl} \vec{F}$  are (relatively) easy to integrate along surfaces. But how do we know if a given vector field is the curl of some vector field? Here is a necessary condition.

**Proposition 4.1.** *Let  $\vec{F}$  be a nice enough vector field (twice continuously differentiable). Then we have  $\operatorname{div}(\operatorname{curl} \vec{F}) \equiv 0$ . In words: a curl is always incompressible.*

*Proof.* Write  $\vec{F} = (F_1, F_2, F_3)$  and abbreviate the partial differentiation operators as

$$\partial_1 = \frac{\partial}{\partial x}, \quad \partial_2 = \frac{\partial}{\partial y}, \quad \partial_3 = \frac{\partial}{\partial z}.$$

Then the curl is

$$\begin{aligned} \operatorname{curl} \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \vec{i}(\partial_2 F_3 - \partial_3 F_2) - \vec{j}(\partial_1 F_3 - \partial_3 F_1) + \vec{k}(\partial_1 F_2 - \partial_2 F_1) \\ &= (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1). \end{aligned}$$

Its divergence is

$$\begin{aligned} \operatorname{div}(\operatorname{curl} \vec{F}) &= \partial_1(\partial_2 F_3 - \partial_3 F_2) + \partial_2(\partial_3 F_1 - \partial_1 F_3) + \partial_3(\partial_1 F_2 - \partial_2 F_1) \\ &= \partial_1 \partial_2 F_3 - \partial_1 \partial_3 F_2 + \partial_2 \partial_3 F_1 - \partial_2 \partial_1 F_3 + \partial_3 \partial_1 F_2 - \partial_3 \partial_2 F_1 \\ &= (\partial_1 \partial_2 F_3 - \partial_2 \partial_1 F_3) + (\partial_2 \partial_3 F_1 - \partial_3 \partial_2 F_1) + (\partial_3 \partial_1 F_2 - \partial_1 \partial_3 F_2) \\ &\equiv 0 + 0 + 0 \\ &= 0. \end{aligned}$$

□



*Remark 4.2.* If one prefers the notation  $\vec{F} = (P, Q, R)$ , then the above calculations can be written as

$$\operatorname{curl} \vec{F} = (R_y - Q_z, P_z - R_x, Q_x - P_y)$$

and

$$\operatorname{div}(\operatorname{curl} \vec{F}) = (R_{yx} - R_{xy}) + (P_{zy} - P_{yz}) + (Q_{xz} - Q_{zx}) \equiv 0.$$

*Example 4.3.* We have seen in section 2 that the vector field  $(-x, 0, z-1)$  is a curl. Its divergence is indeed zero:

$$\operatorname{div}(-x, 0, z-1) \equiv -1 + 0 + 1 = 0.$$

*Example 4.4.* Consider the vector field  $\vec{F} = (xy^2, y+z, yz^3)$ . Is  $\vec{F}$  the curl of some vector field? The divergence is

$$\begin{aligned} \operatorname{div} \vec{F} &= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(y+z) + \frac{\partial}{\partial z}(yz^3) \\ &= y^2 + 1 + 3yz^2 \end{aligned}$$

which is not the constant function 0. Therefore  $\vec{F}$  is **not** the curl of a vector field.

*Question 4.5.* Is the condition also sufficient? In other words, does the property  $\operatorname{div} \vec{F} \equiv 0$  guarantee that  $\vec{F}$  is a curl?

In general, the answer is **NO!** However, there is a partial converse.

**Proposition 4.6.** *Let  $\vec{F}$  be a nice enough vector field **on all of**  $\mathbb{R}^3$ . If  $\vec{F}$  satisfies  $\operatorname{div} \vec{F} \equiv 0$ , then  $\vec{F}$  is the curl of some vector field. In words: a vector field defined everywhere on  $\mathbb{R}^3$  is a curl if and only if it is incompressible.*

*Example 4.7.* Consider the vector field  $\vec{F} = (xy^2, -y^3 + \cos z, 2y^2z)$ . Is  $\vec{F}$  the curl of some vector field?

The divergence is

$$\begin{aligned} \operatorname{div} \vec{F} &= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(-y^3 + \cos z) + \frac{\partial}{\partial z}(2y^2z) \\ &= y^2 - 3y^2 + 2y^2 \\ &\equiv 0. \end{aligned}$$

Moreover,  $\vec{F}$  is defined (and smooth) everywhere on  $\mathbb{R}^3$ . Therefore  $\vec{F}$  **is** the curl of some vector field.

*Remark 4.8.* Describing more precise sufficient conditions for an incompressible vector field to be a curl would require a foray into topology. The answer depends on the shape of the domain of  $\vec{F}$ .

In the lecture on Monday April 30, we will see that the converse of 4.1 can fail spectacularly. We will study a vector field  $\vec{F}$  satisfying  $\operatorname{div} \vec{F} \equiv 0$  which is not a curl.

# Stokes' Theorem Examples

Stokes' Theorem relates surface integrals and line integrals.

## STOKES' THEOREM

Let  $\mathbf{F}$  be a vector field. Let  $S$  be an oriented surface, and let  $C$  be the boundary curve of  $S$ , oriented using the right-hand rule. Then:

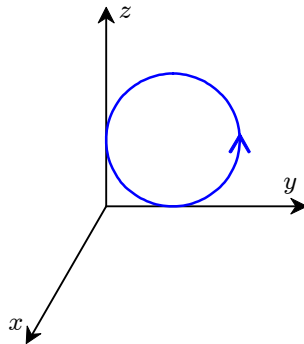
$$\iint_S \text{curl}(\mathbf{F}) \cdot d\mathbf{A} = \oint_C \mathbf{F} \cdot d\mathbf{s}$$

**EXAMPLE 1** Let  $C$  be the curve defined by the parametric equations

$$\begin{aligned}x &= 0 \\y &= 2 + 2 \cos t \quad 0 \leq t \leq 2\pi \\z &= 2 + 2 \sin t\end{aligned}$$

Use Stokes' Theorem to evaluate  $\int_C x^2 e^{5z} dx + x \cos y dy + 3y dz$ .

**SOLUTION** The parametric equations above describe a circle of radius 2 on the  $yz$ -plane:



Let  $D$  be the disc whose boundary is the given circle. By Stokes' Theorem:

$$\int_C (x^2 e^{5z} \mathbf{i} + x \cos y \mathbf{j} + 3y \mathbf{k}) \cdot d\mathbf{s} = \iint_D \text{curl}(x^2 e^{5z} \mathbf{i} + x \cos y \mathbf{j} + 3y \mathbf{k}) \cdot d\mathbf{A}$$

We compute the curl:

$$\text{curl}(x^2 e^{5z} \mathbf{i} + x \cos y \mathbf{j} + 3y \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 e^{5z} & x \cos y & 3y \end{vmatrix} = (3\mathbf{i} + 5x^2 e^{5z} \mathbf{j} + \cos y \mathbf{k})$$

So we have to evaluate  $\iint_D (3\mathbf{i} + 5x^2 e^{5z} \mathbf{j} + \cos y \mathbf{k}) \cdot d\mathbf{A}$ .

This integral can be evaluated geometrically. The vector  $d\mathbf{A}$  for the disc points in the positive  $x$  direction. (Stokes' theorem uses the right-hand rule: if you curl the fingers of your right hand in the direction of  $C$ , then your thumb points in the direction of  $d\mathbf{A}$ .) So:

$$(3\mathbf{i} + 5x^2e^{5z}\mathbf{j} + \cos y\mathbf{k}) \cdot d\mathbf{A} = 3dA$$

Therefore:

$$\iint_D (3\mathbf{i} + 5x^2e^{5z}\mathbf{j} + \cos y\mathbf{k}) \cdot d\mathbf{A} = \iint_D 3dA = 3 \times (\text{the area of the disc}) = 12\pi \quad \blacksquare$$

**EXAMPLE 2** Let  $S$  be the surface  $z = x(1-x)y(1-y)$  for  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ . Evaluate the integral  $\iint_S x\mathbf{k} \cdot d\mathbf{A}$ , where  $d\mathbf{A}$  is the upward-pointing normal vector.

**SOLUTION** If we wish to use Stokes' theorem, we must express  $x\mathbf{k}$  as the curl of some vector field  $\mathbf{F}$ . The formula for the curl is:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

So what we need is:

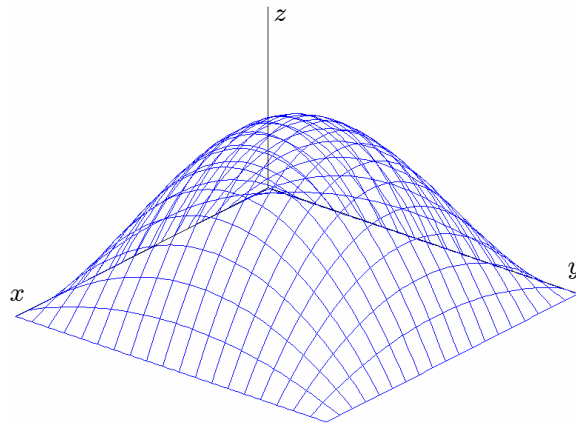
$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0 \quad \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = 0 \quad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = x$$

It is not hard to guess that  $F_y = \frac{1}{2}x^2$ , with  $F_x = F_z = 0$ . Indeed:

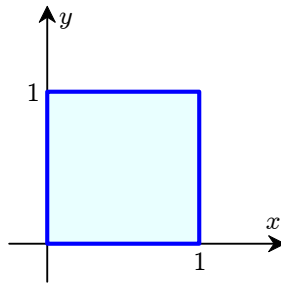
$$\text{curl} \left( \frac{1}{2}x^2\mathbf{j} \right) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & \frac{1}{2}x^2 & 0 \end{vmatrix} = x\mathbf{k}$$

By Stokes' Theorem, we conclude that  $\iint_S x\mathbf{k} \cdot d\mathbf{A} = \oint_C \frac{1}{2}x^2 dy$ , where  $C$  is the boundary curve of the surface  $S$ .

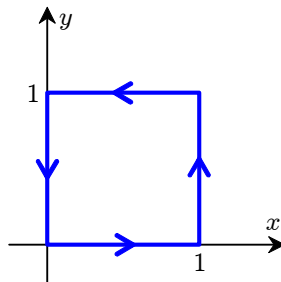
So what is the boundary of  $S$ ? Well, the equation  $z = x(1-x)y(1-y)$  specifies a surface whose  $z$ -coordinate varies with horizontal position:



The allowed values of  $x$  and  $y$  are determined by the inequalities  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ , which describe a square on the  $xy$ -plane:



The sides of this square are along the lines  $x = 0$ ,  $y = 0$ ,  $x = 1$ , and  $y = 1$ . Looking at the equation, we see that  $z = 0$  for these values of  $x$  and  $y$ , so the boundary of the surface is just the boundary of the square on the  $xy$ -plane:



We can evaluate  $\oint_C \frac{1}{2}x^2 dy$  geometrically:

$$\oint_C \frac{1}{2}x^2 dy = \int_{\text{top}} \frac{1}{2}x^2 dy + \int_{\text{bottom}} \frac{1}{2}x^2 dy + \int_{\text{left}} \frac{1}{2}x^2 dy + \int_{\text{right}} \frac{1}{2}x^2 dy$$

The top and bottom sides are horizontal, so  $dy = 0$ . Furthermore,  $x = 0$  along the left edge, and  $x = 1$

along the right edge, so:

$$\oint_C \frac{1}{2}x^2 dy = 0 + 0 + 0 + \int_{\text{right}} \frac{1}{2} dy = \frac{1}{2} \quad \blacksquare$$

**EXAMPLE 3** Let  $S$  be the upper hemisphere of the unit sphere  $x^2 + y^2 + z^2 = 1$ . Use Stokes' theorem to evaluate  $\iint_S (x^3e^y \mathbf{i} - 3x^2e^y \mathbf{j}) \cdot d\mathbf{A}$ , where  $d\mathbf{A}$  is the upward-pointing normal vector.

**SOLUTION** If we wish to use Stokes' theorem, we must express  $x^3e^y \mathbf{i} - 3x^2e^y \mathbf{j}$  as the curl of some vector field  $\mathbf{F}$ . The formula for the curl is:

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

So what we need is:

$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = x^3e^y \quad \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = -3x^2e^y \quad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0$$

If we guess that  $F_y$  and  $F_x$  are zero, it is not too hard to figure out that  $F_z = x^3e^y$ . Indeed:

$$\text{curl}(x^3e^y \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & x^3e^y \end{vmatrix} = x^3e^y \mathbf{i} - 3x^2e^y \mathbf{j}$$

By Stokes' Theorem, we conclude that  $\iint_S (x^3e^y \mathbf{i} - 3x^2e^y \mathbf{j}) \cdot d\mathbf{A} = \oint_C x^3e^y dz$ , where  $C$  is the boundary curve of the surface  $S$ .

Since  $S$  is the upper hemisphere of the unit sphere,  $C$  is just the unit circle on the  $xy$ -plane. By the right-hand rule,  $C$  is oriented counterclockwise. (It would be clockwise if we had started with the downward-pointing  $d\mathbf{A}$ .) Then  $\oint_C x^3e^y dz$  is zero, since  $z$  is not changing over the course of the circle.

We conclude that  $\iint_S (x^3e^y \mathbf{i} - 3x^2e^y \mathbf{j}) \cdot d\mathbf{A} = 0$ . ■

**EXAMPLE 4** Let  $S$  be the surface defined by  $z = x^2 + y^2$  for  $z \leq 4$ . Use Stokes' Theorem to evaluate  $\iint_S (-3xz^2 \mathbf{i} + z^3 \mathbf{k}) \cdot d\mathbf{A}$ , where  $d\mathbf{A}$  is the upward-pointing normal vector.

**SOLUTION** If we wish to use Stokes' theorem, we must express  $-3xz^2 \mathbf{i} + z^3 \mathbf{k}$  as the curl of some vector field  $\mathbf{F}$ . The formula for the curl is:

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

So what we need is:

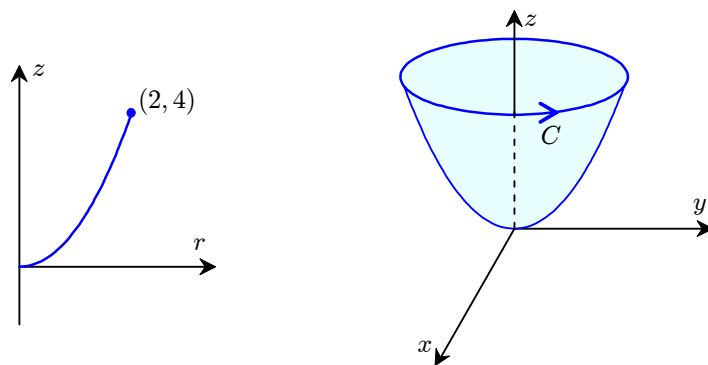
$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = -3xz^2 \quad \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = 0 \quad \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = z^3$$

If we guess that  $F_x$  and  $F_z$  are zero, it is not hard to figure out that  $F_y = xz^3$ . Indeed:

$$\operatorname{curl}(xz^3 \mathbf{j}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & xz^3 & 0 \end{vmatrix} = -3xz^2 \mathbf{i} + z^3 \mathbf{k}$$

By Stokes' Theorem, we conclude that  $\iint_S (-3xz^2 \mathbf{i} + z^3 \mathbf{k}) \cdot d\mathbf{A} = \oint_C xz^3 dy$ , where  $C$  is the boundary curve of the surface  $S$ .

So what is the boundary of  $S$ ? Well, the equation for the surface  $S$  can be expressed as  $z = r^2$ , where  $z \leq 4$ . This appears as a parabola on the  $rz$ -plane:



Therefore, the surface  $S$  is a bowl-shaped paraboloid. Its boundary curve  $C$  is a counterclockwise circle on the plane  $z = 4$  with radius 2, with parameterization:

$$\begin{aligned} x &= 2 \cos t \\ y &= 2 \sin t \quad 0 \leq t \leq 2\pi \\ z &= 4 \end{aligned}$$

(The circle is counterclockwise by the right-hand rule.) Thus:

$$\begin{aligned}\oint_C xz^3 dy &= \int_0^{2\pi} (2 \cos t)(4)^3(2 \cos t dt) \\ &= 256 \int_0^{2\pi} \cos^2 t dt \\ &= 256 \int_0^{2\pi} \frac{1}{2}(1 + \cos 2t) dt \quad (\text{double-angle formula}) \\ &= 128 \left[ t + \frac{1}{2} \sin 2t \right]_0^{2\pi} \\ &= 256\pi\end{aligned}$$

