Homework 11, Math 250-07

due Wednesday, November 28, 2012

This home covers material on Green functions and simple linear systems.

1. Consider $y'' + 6y' + 9y = f(x)$.

   (a) Find the Green function for this equation.
   Answer: *We solve $y'' + 6y' + 9y = \delta(x)$, $y(0) = y'(0) = 0$ using Laplace transforms, and find the one-sided Green function*

   $$g(x) = u_0(x)xe^{-3x}.$$

   (b) Find a particular solution (in integral form) to this equation when $f(x) = x + e^{-3x}$.
   Answer: *Using the Green function, we find a particular solution*

   $$y_p(x) = \int_0^x (x - v + e^{-3(x-v)})ve^{-3v}dv$$

   (c) Find the general solution when $f(x) = \dfrac{\sin(2x)}{x}$.
   Answer: *The general solution is the sum of the homogeneous solution and any particular solution. We can find the homogeneous solution using our usual characteristic equation, and we can find a particular solution using the Green function.*

   $$y(x) = C_1e^{-3x} + C_2xe^{-3x} + \int_0^x \frac{\sin(2(x-v))}{x-v}ve^{-3v}dv$$

2. Suppose $x(t)$ and $y(t)$ are functions satisfying the system of linear equations

   $$\dot{x} + \dot{y} = 3x - 2y, \quad \dot{x} - 2\dot{y} = 3x + 4y.$$

   Find the general solutions for $x(t)$ and $y(t)$.
   Answer: *Using some algebra, we find*

   $$\dot{x} = 3x, \quad \dot{y} = -2y$$

   These we can solve using integration.

   $$x(t) = C_1e^{3t}, \quad y(t) = C_2e^{-2t}$$

3. Suppose $x(t)$ and $y(t)$ are functions satisfying the system of linear equations

   $$\dot{x} = 6x - y, \quad \dot{y} = 5y.$$

   Find the general solutions for $x(t)$ and $y(t)$.
   Answer: *First we solve the y equation, then we substitute and solve the x equation.*

   $$y(t) = C_1e^{5t}, \quad x(t) = C_2e^{6t} + C_1e^{5t}$$

4. Consider the 2-dimensional linear system

   $$\dot{x} = x + 4y, \quad \dot{y} = 2x - 6y.$$

   We'll solve this the old-fashioned way.

   (a) Let $u = x - 4y$ and $v = 2x + y$. Find $x(u, v)$ and $y(u, v)$. (In linear algebra, this is called a change-of-basis.)
   Answer:

   $$x = \frac{u + 4v}{9}, \quad y = \frac{-2u + v}{9}$$

(b) Determine equations for $\dot{u}$ and $\dot{v}$ in terms of u and v.
Answer: $\dot{u} = -7u,\ \dot{v} = 2v.$

(c) Find the general solution for $u(t)$ and $v(t)$.
Answer:

$$u(t) = C_1 e^{-7t}, \quad v(t) = C_2 e^{2t}$$

(d) Find the general solution for $x(t)$ and $y(t)$.
Answer:

$$x = \frac{C_1 e^{-7t} + 4C_2 e^{2t}}{9}, \quad y = \frac{-2C_1 e^{-7t} + C_2 e^{2t}}{9}$$

or

$$x = C_1 e^{-7t} + 4C_2 e^{2t}, \quad y = -2C_1 e^{-7t} + C_2 e^{2t}$$