1. Számoljuk ki az alábbi függvény Foirier-transzformáltját!

$$f(t) = \begin{cases} 0 & \text{ha } x < 0 \\ \sin(t) & \text{ha } x > 0 \end{cases} \tag{1}$$

Azaz $f(t) = \theta(t)\sin(t)$.

Definíció szerint:

$$\begin{aligned} 2\pi\tilde{F} &= \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \int_0^{\infty} \frac{e^{it} - e^{-it}}{2i} e^{-i\omega t}dt = \\ &= \frac{1}{2i}\int_0^{\infty} e^{it-i\omega t} - e^{-it-i\omega t}dt = \\ &= \frac{1}{2i}\int_0^{\infty} e^{-it(\omega-1)} - e^{-it(\omega+1)}dt \end{aligned} \tag{2}$$

Az alsó határ egyértelmű, a felső határ egy fél Dirac-delta, azaz:

$$\tilde{F} = \frac{1}{2\pi(1-\omega^2)} + \frac{i}{4}\left[\delta(\omega-1) - \delta(\omega+1)\right] \tag{3}$$

Konvolúcióval:

$$h(t)g(t) = \int_{-\infty}^{\infty} \tilde{H}(\omega')\tilde{G}(\omega-\omega')d\omega' \tag{4}$$

$$\begin{aligned} \mathcal{F}[\theta(t)] &= \frac{1}{2}\left(\frac{1}{i\pi\omega} + \delta(\omega)\right) \\ \mathcal{F}[\sin(t)] &= \frac{1}{2}\left(\delta(\omega-1) + \delta(\omega+1)\right) \end{aligned} \tag{5}$$

Most:

$$\begin{aligned} \theta(t)\sin(t) &= \frac{1}{4}\int_{-\infty}^{\infty}\left(\frac{1}{i\pi(\omega-\omega')} + \delta(\omega-\omega')\right)\left(\delta(\omega'-1) + \delta(\omega'+1)\right)d\omega' = \\ &= \frac{1}{2\pi}\left(\frac{1}{1-\omega} + \frac{1}{\omega+1} + \frac{\pi i}{2}[\delta(\omega-1) - \delta(\omega+1)]\right) \end{aligned} \tag{6}$$

Ami ugyanaz, mint az előző.

2. Keressük meg az alábbi differenciálegyenlet Green-függvényét!

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1. Find the 1-dimensional Green's function for the interval $(0, l)$. The three properties defining it can be stated as follows:

   (a) It solves $G''(x) = 0$ for $x \neq x_0$.

   (b) $G(0) = G(l) = 0$.

   (c) $G(x)$ is continuous at x_0 and $G(x) + \frac{1}{2}|x - x_0|$ is harmonic at x_0.

   The first property gives us that

$$G(x) = \begin{cases} a_0 x + b_0 & 0 < x < x_0 \\ a_1 x + b_1 & x_0 < x < l \end{cases}$$

   Now we apply $G(0) = G(1) = 0$ and the fact that G is continuous at x_0 to get

$$G(x) = \begin{cases} ax & 0 < x < x_0 \\ \frac{a x_0}{x_0 - l}(x - l) & x_0 < x < 1 \end{cases}$$

   We just need to use the fact that $R(x) = G(x) + \frac{1}{2}|x - x_0|$ is harmonic at x_0 to find a. The fact that this function is harmonic implies that its first derivative is continuous at x_0. A simple calculation gives:

$$\lim_{x \to x_0-} R'(x) = a - \frac{1}{2}$$

   and

$$\lim_{x \to x_0+} R'(x) = \frac{a x_0}{x_0 - l} + \frac{1}{2}$$

   Equating these limits results in

$$a = 1 - \frac{x_0}{l}$$

   So

$$G(x) = \begin{cases} \left(1 - \frac{x_0}{l}\right) x & 0 < x < x_0 \\ -\frac{x_0}{l}(x - l) & x_0 < x < l \end{cases}$$

3. Oldjuk meg az alábbi feladatot! Kiegészítések a rövid megoldásokhoz:

   - 1.a. Itt mind a két gyök -3 ezért van szükség az $c_1 x \exp(-3x) + c_2 \exp(-3x)$ gyökökre. A peremfeltételek miatt azonban csak az $x \exp(-3x)$ tag marad.
   - 1.b. Ez ugye a szimpla konvolúció a megoldás meghatározására

$$\int_0^x \left(x - v + e^{-3(x-v)}\right) v e^{-3v} dv = \frac{1}{54}\left[e^{-3x}(4 + 6x + 27x^2) + (6x - 4)\right] \tag{7}$$

   - 2.c. Ezt nem lehet szépen kiintegrálni.

   A feladat:

1. Consider $y'' + 6y' + 9y = f(x)$.

   (a) Find the Green function for this equation.
   Answer: *We solve $y'' + 6y' + 9y = \delta(x)$, $y(0) = y'(0) = 0$ using Laplace transforms, and find the one-sided Green function*
   $$g(x) = u_0(x)xe^{-3x}.$$

   (b) Find a particular solution (in integral form) to this equation when $f(x) = x + e^{-3x}$.
   Answer: *Using the Green function, we find a particular solution*
   $$y_p(x) = \int_0^x (x - v + e^{-3(x-v)})ve^{-3v}dv$$

   (c) Find the general solution when $f(x) = \dfrac{\sin(2x)}{x}$.
   Answer: *The general solution is the sum of the homogeneous solution and any particular solution. We can find the homogeneous solution using our usual characteristic equation, and we can find a particular solution using the Green function.*
   $$y(x) = C_1e^{-3x} + C_2xe^{-3x} + \int_0^x \frac{\sin(2(x-v))}{x-v}ve^{-3v}dv$$

4. Oldjuk meg az alábbi differenciálegyenlet rendszert!

   3. Suppose $x(t)$ and $y(t)$ are functions satisfying the system of linear equations
   $$\dot{x} = 6x - y, \quad \dot{y} = 5y.$$
   Find the general solutions for $x(t)$ and $y(t)$.
   Answer: *First we solve the y equation, then we substitute and solve the x equation.*
   $$y(t) = C_1e^{5t}, \quad x(t) = C_2e^{6t} + C_1e^{5t}$$

5. Oldjuk meg az alábbi differenciálegyenlet rendszert!

   4. Consider the 2-dimensional linear system
   $$\dot{x} = x + 4y, \quad \dot{y} = 2x - 6y.$$
   We'll solve this the old-fashioned way.

   (a) Let $u = x - 4y$ and $v = 2x + y$. Find $x(u, v)$ and $y(u, v)$. (In linear algebra, this is called a change-of-basis.)
   Answer:
   $$x = \frac{u + 4v}{9}, \quad y = \frac{-2u + v}{9}$$

(b) Determine equations for $\dot{u}$ and $\dot{v}$ in terms of u and v.
Answer: $\dot{u} = -7u$, $\dot{v} = 2v$.

(c) Find the general solution for $u(t)$ and $v(t)$.
Answer:
$$u(t) = C_1 e^{-7t}, \quad v(t) = C_2 e^{2t}$$

(d) Find the general solution for $x(t)$ and $y(t)$.
Answer:
$$x = \frac{C_1 e^{-7t} + 4C_2 e^{2t}}{9}, \quad y = \frac{-2C_1 e^{-7t} + C_2 e^{2t}}{9}$$
or
$$x = C_1 e^{-7t} + 4C_2 e^{2t}, \quad y = -2C_1 e^{-7t} + C_2 e^{2t}$$

6. A honlapon `http://www.phy.bme.hu/~vektor/2015tavasz/` a 10. pont Inhomogén másodrendű differenciálegyenletek dokumentum 4. feladata, de lehet kezdeni a 3-assal.