2. Gyakorlat

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1. Adjuk meg a következő differenciálegyenlet általános megoldását:

$$\begin{aligned} &\text{a.)} \quad y' = \ln(x)y \\ &\text{b.)} \quad y' - \frac{2y}{x} = 0 \end{aligned}$$

2. An integrating factor is any expression that a differential equation is multiplied by to facilitate integration and is not restricted to first order linear equations. For example, the nonlinear second order equation

$$\frac{d^2y}{dt^2} = Ay^{2/3}$$

admits $\frac{dy}{dt}$ as an integrating factor:

$$\frac{d^2y}{dt^2}\frac{dy}{dt} = Ay^{2/3}\frac{dy}{dt}.$$

To integrate, note that both sides of the equation may be expressed as derivatives by going backwards with the [[chain rule]]:

$$\frac{d}{dt}\left(\frac{1}{2}\left(\frac{dy}{dt}\right)^2\right) = \frac{d}{dt}\left(A\frac{3}{5}y^{5/3}\right).$$

Therefore

$$\left(\frac{dy}{dt}\right)^2 = \frac{6A}{5}y^{5/3} + C_0.$$

This form may be more useful, depending on application. Performing a separation of variables will give:

$$\int \frac{dy}{\sqrt{\frac{6A}{5}y^{5/3} + C_0}} = t + C_1;$$

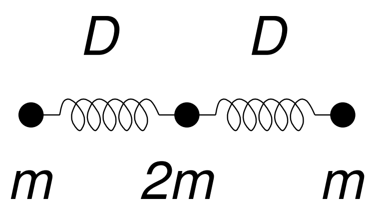
this is an implicit solution which involves a nonelementary integral. Though likely too obscure to be useful, this is a general solution. Also, because the previous equation is first order, it could be used for numeric solution in favor of the original equation.

3. Bolygómozgás
   Egy rendszer energiáját 2D polárkoordinátákban a következőképpen tudjuk megadni:

$$E = \frac{1}{2}\frac{l^2}{mr^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{2}\frac{l^2}{mr^2} - \frac{\alpha}{r}\ .$$

   Határozzuk meg a rendszer $r(\varphi)$ pályáját!

4. Az ábrán látható m tömegű testeket jobbra, balra meghúzzuk d-vel, majd elegendjük. írjuk le a három test mozgását! Mik lesznek a karakterisztikus frekvenciák?



The Characteristic Polynomial

Back to the subject of the second order linear homogeneous equations with constant coefficients (note that it is not in the standard form below):

$$a y'' + b y' + c y = 0, \qquad a \neq 0. \qquad (*)$$

We have seen a few examples of such an equation. In all cases the solutions consist of exponential functions, or terms that could be rewritten into exponential functions[†]. With this fact in mind, let us derive a (very simple, as it turns out) method to solve equations of this type. We will start with the assumption that there are indeed some exponential functions of unknown exponents that would satisfy any equation of the above form. We will then devise a way to find the specific exponents that would give us the solution.

Let $y = e^{rt}$ be a solution of (*), for some as-yet-unknown constant r. Substitute y, $y' = re^{rt}$, and $y'' = r^2 e^{rt}$ into (*), we get

$$a r^2 e^{rt} + b r e^{rt} + c e^{rt} = 0, \qquad \text{or}$$

$$e^{rt}(a r^2 + b r + c) = 0.$$

Since e^{rt} is never zero, the above equation is satisfied (and therefore $y = e^{rt}$ is a solution of (*)) if and only if $a r^2 + b r + c = 0$. Notice that the expression $a r^2 + b r + c$ is a quadratic polynomial with r as the unknown. It is always solvable, with roots given by the quadratic formula. Hence, we can always solve a second order linear homogeneous equation with constant coefficients (*).

[†] Sine and cosine are related to exponential functions by the identities

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

This polynomial, $a r^2 + b r + c$, is called the *characteristic polynomial* of the differential equation (*). The equation

$$\boxed{a r^2 + b r + c = 0}$$

is called the *characteristic equation* of (*). Each and every root, sometimes called a *characteristic root*, r, of the characteristic polynomial gives rise to a solution $y = e^{rt}$ of (*).

We will take a more detailed look of the 3 possible cases of the solutions thusly found:

1. (When $b^2 - 4ac > 0$) There are two distinct real roots r_1, r_2.
2. (When $b^2 - 4ac < 0$) There are two complex conjugate roots $r = \lambda \pm \mu i$.
3. (When $b^2 - 4ac = 0$) There is one repeated real root r.

Note: There is no need to put the equation in its standard form when solving it using the characteristic equation method. The roots of the characteristic equation remain the same regardless whether the leading coefficient is 1 or not.

Case 1 Two distinct real roots

When $b^2 - 4ac > 0$, the characteristic polynomial have two distinct real roots r_1, r_2. They give two distinct[‡] solutions $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$. Therefore, a general solution of (*) is

$$\boxed{y = C_1 y_1 + C_2 y_2 = C_1 e^{r_1 t} + C_2 e^{r_2 t}.}$$

It is **that** easy.

Example: $$y'' + 5y' + 4y = 0$$

The characteristic equation is $r^2 + 5r + 4 = (r + 1)(r + 4) = 0$, the roots of the polynomial are $r = -1$ and -4. The general solution is then

$$y = C_1 e^{-t} + C_2 e^{-4t}.$$

Suppose there are initial conditions $y(0) = 1$, $y'(0) = -7$. A unique particular solution can be found by solving for C_1 and C_2 using the initial conditions. First we need to calculate $y' = -C_1 e^{-t} - 4C_2 e^{-4t}$, then apply the initial values:

$$1 = y(0) = C_1 e^0 + C_2 e^0 = C_1 + C_2$$

$$-7 = y'(0) = -C_1 e^0 - 4C_2 e^0 = -C_1 - 4C_2$$

The solution is $C_1 = -1$, and $C_2 = 2$ $\quad\rightarrow\quad$ $y = -e^{-t} + 2e^{-4t}.$

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‡ We shall see the precise meaning of distinctness in the next section. For now just think that the two solutions are not constant multiples of each other.

Question: Suppose the initial conditions are instead $y(10000) = 1$, $y'(10000) = -7$. How would the new t_0 change the particular solution?

Apply the initial conditions as before, and we see there is a little complication. Namely, the simultaneous system of 2 equations that we have to solve in order to find C_1 and C_2 now comes with rather inconvenient irrational coefficients:

$$1 = y(10000) = C_1 e^{-10000} + C_2 e^{-40000}$$

$$-7 = y'(10000) = -C_1 e^{-10000} - 4C_2 e^{-40000}$$

With some good bookkeeping, systems like this can be solved the usual way. However, there is an easier method to simplify the inconvenient coefficients. The idea is *translation* (or *time-shift*). What we will do is to first construct a new coordinate axis, say $\check{T}$-axis. The two coordinate-axes are related by the equation $\check{T} = t - t_0$. (Therefore, when $t = t_0$, $\check{T} = 0$; that is, the initial t-value t_0 becomes the new origin.) In other words, we translate (or time-shift) t-axis by t_0 units to make it $\check{T}$-axis. In this example, we will accordingly set $\check{T} = t - 10000$. The immediate effect is that it makes the initial conditions to be back at 0: $y(0) = 1$, $y'(0) = -7$, with respect to the new $\check{T}$-coordinate. We then solve the translated system of 2 equations to find C_1 and C_2. What we get is the (simpler) system

$$1 = y(0) = C_1 e^{0} + C_2 e^{0} = C_1 + C_2$$

$$-7 = y'(0) = -C_1 e^{0} - 4C_2 e^{0} = -C_1 - 4C_2$$

As we have seen on the previous page, the solution is $C_1 = -1$, and $C_2 = 2$. Hence, the solution, in the new $\check{T}$-coordinate system, is $y(\check{T}) = -e^{-\check{T}} + 2e^{-4\check{T}}$.

Lastly, since this solution is in terms of $\check{T}$, but the original problem was in terms of t, we should convert it back to the original context. This conversion is easily achieved using the translation formula used earlier, $\check{T} = t - t_0 = t - 10000$. By replacing every occurrence of $\check{T}$ by $t - 1000$ in the solution, we obtain the solution, in its proper independent variable t.

$$y(t) = -e^{-(t - 10000)} + 2e^{-4(t - 10000)}.$$

Example: Consider the solution $y(t)$ of the initial value problem

$$y'' - 2y' - 8y = 0, \qquad y(0) = \alpha, \quad y'(0) = 2\pi.$$

Depending on the value of α, as $t \to \infty$, there are 3 possible behaviors of $y(t)$. Explicitly determine the possible behaviors and the respective initial value α associated with each behavior.

The characteristic equation is $r^2 - 2r - 8 = (r + 2)(r - 4) = 0$. Its roots are $r = -2$ and 4. The general solution is then

$$y = C_1 e^{-2t} + C_2 e^{4t}.$$

Notice that the long-term behavior of the solution is dependent on the coefficient C_2 only, since the $C_1 e^{-2t}$ term tends to 0 as $t \to \infty$, regardless of the value of C_1.

Solving for C_2 in terms of α, we get

$$\begin{aligned} y(0) = \alpha &= C_1 + C_2 \\ y'(0) = 2\pi &= -2C_1 + 4C_2 \end{aligned}$$

$$2\alpha + 2\pi = 6C_2 \qquad \to \qquad C_2 = \frac{\alpha + \pi}{3}.$$

Now, if $C_2 > 0$ then y tends to ∞ as $t \to \infty$. This would happen when $\alpha > -\pi$. If $C_2 = 0$ then y tends to 0 as $t \to \infty$. This would happen when $\alpha = -\pi$. Lastly, if $C_2 < 0$ then y tends to $-\infty$ as $t \to \infty$. This would happen when $\alpha < -\pi$. In summary:

When $\alpha > -\pi$,	$C_2 > 0$,	$\lim_{t\to\infty} y(t) = \infty$.
When $\alpha = -\pi$,	$C_2 = 0$,	$\lim_{t\to\infty} y(t) = 0$.
When $\alpha < -\pi$,	$C_2 < 0$,	$\lim_{t\to\infty} y(t) = -\infty$.