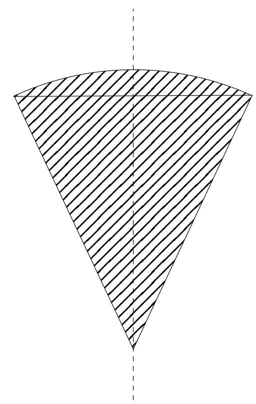
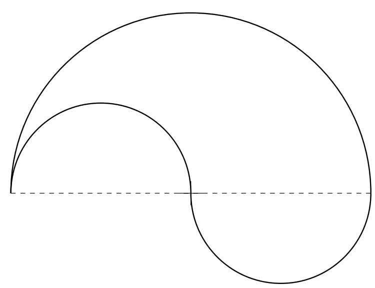
2. Gyakorlat

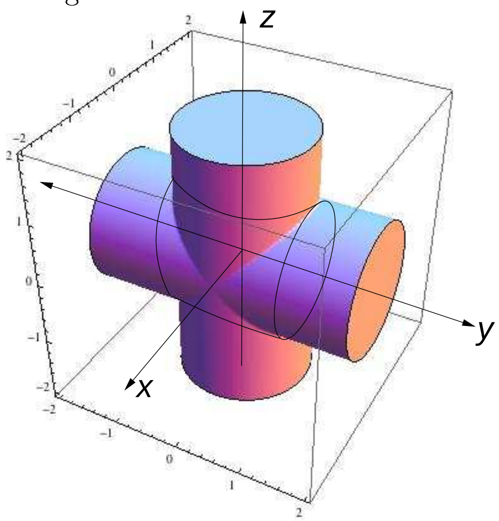
1. Határozzuk meg az ábrán látható, α szögű kőrcikk területét, tömegközéppontját és tehetetlenségi nyomatékát a szaggatot vonallal jelölt tengelyre!



2. Határozzuk meg az alábbi alakzat területét és tömegközéppontját (ha van idő, akkor a tehetetlenségi nyomatékát a szaggatot vonallal rajzolt tengelyre)!



3. Határozzuk meg két R sugarú, egymásra merőleges tengelyű henger közös részének a térfogatát!



A hengerek tengelyei essenek egybe az y és z temgelyekkel. Ebben az esetben a két henger egyenlete a következő lesz:

$$R^2 = x^2 + y^2, \qquad R^2 = x^2 + z^2$$

A két henger közös részére a következő két feltételnek kell teljesülnie.

$$x^2 + y^2 \leq R^2, \qquad x^2 + z^2 \leq R^2$$

A térfogat meghatározásához integráljuk az $f(x, y, z) \equiv 1$ függvényt!

Rögzítsük x és y értékét úgy, hogy eleget tegyenek az előző feltételeknek és integráljunk z szerint:

$$I_1(x, y) = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dz = 2\sqrt{R^2 - x^2}$$

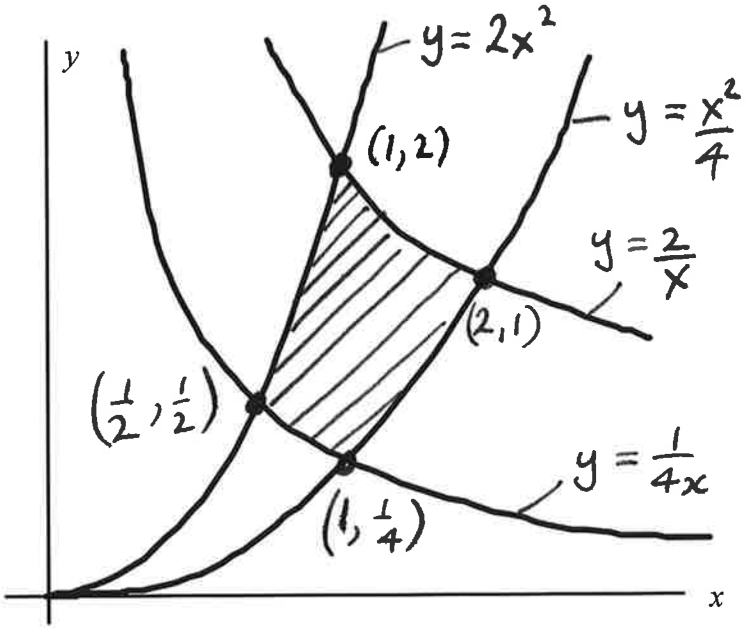
Második lépésként integráljuk az $I_1(x, y)$ függvényt az y változó szerint figyelembe véve a határokra vonatkozó feltételt:

$$I_2(x) = \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2\sqrt{R^2 - x^2} dy = 4\left(R^2 - x^2\right)$$

Végül integráljunk $-R$-től R-ig:

$$V = \int_{-R}^{R} 4\left(R^2 - x^2\right) dx = \frac{16}{3} R^3$$

B) Find the area of the region internal to the lines $y = 2x^2$, $y = x^2/4$, $y = 2/x$ and $y = 1/(4x)$.



$$I = \iint_D dx\,dy$$

Any change of variable that made the specification of the region tractable would be a big help. We make the substitution:

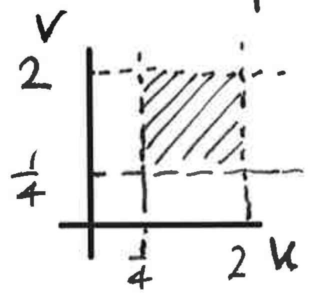
$$u = \frac{y}{x^2} \qquad v = xy$$

$$J^1 = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ y & x \end{vmatrix}$$

$$= \left|-\frac{3y}{x^2}\right| = 3u$$

$$\Rightarrow \quad J = \frac{1}{3u}$$

Domain in u,v space

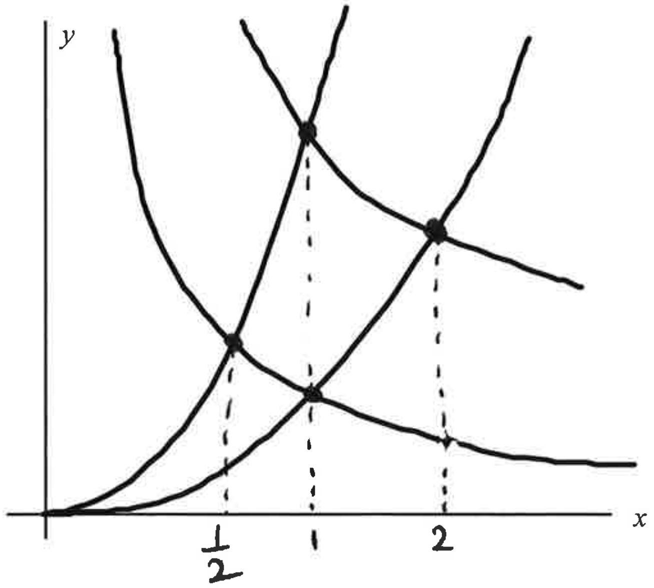


The required area is thus

$$\int_{v=1/4}^{2}\int_{u=1/4}^{2} \frac{1}{3u}\,du\,dv = \frac{1}{3}\int_{v=1/4}^{2} [\ln u]_{1/4}^{2}\,dv = \frac{1}{3}\int_{v=1/4}^{2}\left(\ln 2 - \ln\frac{1}{4}\right)dv$$

$$= \frac{\ln 8}{3}[v]_{1/4}^{2} = \frac{\ln 8}{3}\left(2 - \frac{1}{4}\right) = \frac{7}{12}\ln 2^3 = \frac{7}{4}\ln 2$$

It is, of course, possible to do this question in the original variables but doing so involves a far more tricky piece of manipulation



$$\text{Area} = \int_{x=\frac{1}{2}}^{1}\int_{y=\frac{1}{4x}}^{2x^2} dy\,dx + \int_{x=1}^{2}\int_{y=\frac{x^2}{4}}^{\frac{2}{x}} dy\,dx$$

$$= \int_{x=\frac{1}{2}}^{1}\left[y\right]_{\frac{1}{4x}}^{2x^2} dx + \int_{x=1}^{2}\left[y\right]_{\frac{x^2}{4}}^{\frac{2}{x}} dx$$

$$= \int_{x=1/2}^{1}\left(2x^2 - \frac{1}{4x}\right)dx + \int_{x=1}^{2}\left(\frac{2}{x} - \frac{x^2}{4}\right)dx$$

$$= \left[2\frac{x^3}{3} - \frac{1}{4}\ln x\right]_{1/2}^{1} + \left[2\ln x - \frac{x^3}{12}\right]_{1}^{2}$$

$$= \frac{2}{3} - \frac{1}{4}\ln 1 - \left(\frac{1}{12} - \frac{1}{4}\ln\frac{1}{2}\right) + 2\ln 2 - \frac{2}{3} - \left(2\ln 1 - \frac{1}{12}\right)$$

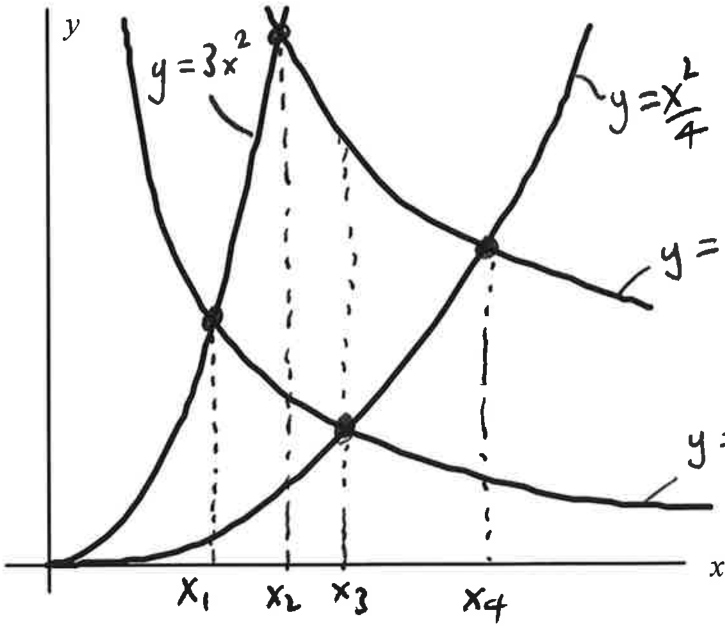
$$= 2\ln 2 - \frac{1}{4}\ln 2 = \frac{7}{4}\ln 2$$

Note that the curves were chosen to be kind here. If the region had been

$$y = 3x^2,\quad y = x^2/4,\quad y = 2/x \quad\text{and}\quad y = 1/(4x)$$

this problem would have involved *three* regions.

only change when use u & v is upper limit for u changes $2 \to 3$

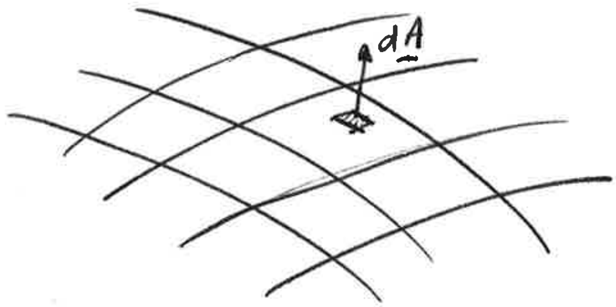


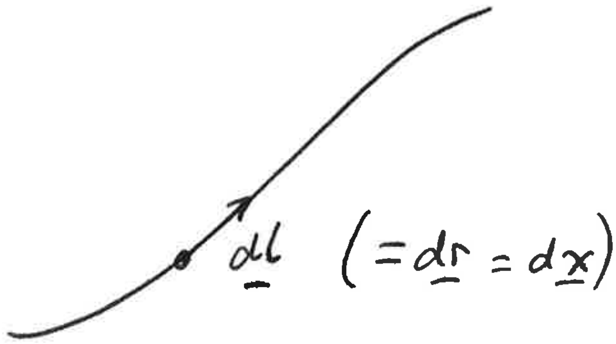
$$\text{Area} = \int_{x_1}^{x_2}\int_{y=\frac{1}{4x}}^{3x^2} dy\,dx + \int_{x_2}^{x_3}\int_{y=\frac{1}{4x}}^{2/x} dy\,dx + \int_{x_3}^{x_4}\int_{y=\frac{x^2}{4}}^{\frac{2}{x}} dy\,dx$$

Can now do Ex 2 Q3

4.2 Surface and line integrals

The thing which makes 3D surface and line integrals more difficult than the examples we have just been doing is that the integration *element* is now a vector: the area element $\underline{dA}$ or the line element $\underline{dl}$.





$$\underline{dA} = dA\,\underline{\hat{n}}$$

If you integrate in the opposite direction along a curve, $\underline{dl}$ changes sign so the integral changes sign. With area elements, there is an ambiguity about which direction to choose for $\underline{dA}$: upwards or downwards? For a surface with a boundary curve, it is important to define what convention is being used. For a *closed* surface with no boundary curve, the convention is to choose the *outward-pointing* normal, relative to the volume enclosed within the surface.