(Szá Má Ism. 1. 2. ZH. MINTAMEGOLDÁS (−1−) 2014. Nov.

1,

$$\det A = \begin{vmatrix} -2 & 3 & 4 \\ -1 & 3 & -5 \\ 3 & -2 & 1 \end{vmatrix} = \underbrace{(-6-45+8)}_{-43} - \underbrace{(36-3-20)}_{13} = -56$$

$\det A \neq 0$, így $\exists A^{-1}$, és $\operatorname{rank} A = 3$.

$\det B = 0$, mert a 3. sor az első kettő összege. Így B nem invertálható. Az első két sor lineárisan független, így $\operatorname{rank} B = 2$.

— ∘ —

2,

$$\left[\begin{array}{cccc|c} 2 & 1 & 1 & -3 & 5 \\ 2 & 0 & 2 & 0 & 6 \\ 0 & -1 & 3 & 3 & 3 \\ 1 & 2 & -3 & -6 & -1 \end{array}\right] \begin{array}{l} \times(-2) \\ \times(-2) \\ \\ \\ \end{array} \Rightarrow \left[\begin{array}{cccc|c} 0 & -3 & 7 & 9 & 7 \\ 0 & -4 & 8 & 12 & 8 \\ 0 & -1 & 3 & 3 & 3 \\ 1 & 2 & -3 & -6 & -1 \end{array}\right] \begin{array}{l} \\ \times\frac{1}{4} \\ \times(-1) \\ \\ \end{array} \Rightarrow$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -3 & -6 & -1 \\ 0 & 1 & -3 & -3 & -3 \\ 0 & -1 & 2 & 3 & 2 \\ 0 & -3 & 7 & 9 & 7 \end{array}\right] \begin{array}{l} \\ \times 1 \\ \times 3 \\ \\ \end{array} \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -3 & -6 & -1 \\ 0 & 1 & -3 & -3 & -3 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -2 & 0 & -2 \end{array}\right] \begin{array}{l} \\ \\ \times(-1) \\ \times 2 \\ \end{array} \Rightarrow$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -3 & -6 & -1 \\ 0 & 1 & -3 & -3 & -3 \\ 0 & 0 & 1 & 0 & 1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{array}\right] \begin{array}{l} \times 3 \\ \times 3 \\ \\ \\ \end{array}$$

Innen látható, hogy $\operatorname{rank} A = 3$, $x_4 \in \mathbb{R}$ tetszőleges; $x_3 = 1$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & -6 & 2 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right] \times(-2) \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right]$$

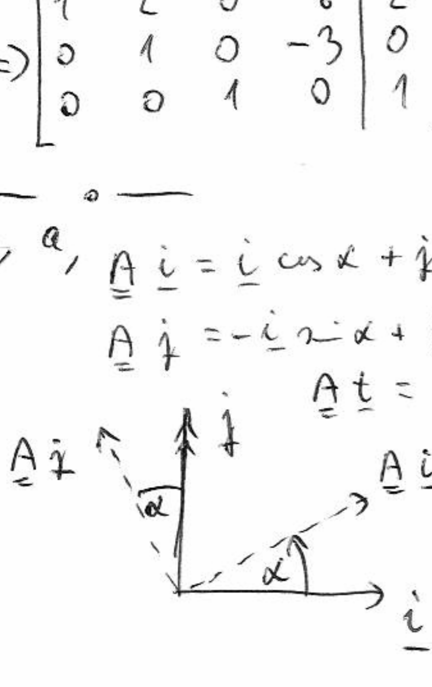
$x_2 = -3x_4$

$x_1 = 2$

— ∘ —

3, a,

$$\left.\begin{array}{l} \underline{\underline{A}}\,\underline{i} = \underline{i}\cos\alpha + \underline{j}\sin\alpha \\ \underline{\underline{A}}\,\underline{j} = -\underline{i}\sin\alpha + \underline{j}\cos\alpha \\ \underline{\underline{A}}\,\underline{k} = \underline{k} \end{array}\right\} \Rightarrow [\underline{\underline{A}}]^B = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



az együtthatók a mátrix oszlopai.

b, $\mathcal{B}=\{\underline{i},\underline{j},\underline{t}\}$; $\mathcal{K}=\{\underline{i},\underline{j},\underline{k}\}$ $\underline{\underline{F}}\,\underline{i}=\underline{i}$, $\underline{\underline{F}}\,\underline{j}=\underline{j}$, $\underline{\underline{F}}\,\underline{t}=\underline{k}$

$\underline{t}=\underline{i}+2\underline{k}$, tehát $\underline{k}=\frac{1}{2}\underline{t}-\frac{1}{2}\underline{i}$, tehát $\underline{\underline{F}}\,\underline{t}=-\frac{1}{2}\underline{i}+\frac{1}{2}\underline{t}$. Így

$$[\underline{\underline{F}}]^{\mathcal{B}}=\begin{bmatrix}1 & 0 & -1/2\\ 0 & 1 & 0\\ 0 & 0 & 1/2\end{bmatrix}$$

De számolhatjuk $[\underline{\underline{F}}]^{\mathcal{K}}$-t is; $\underline{\underline{F}}\,\underline{t}=\underline{\underline{F}}(\underline{i}+2\underline{k})=$

$$=\underbrace{\underline{\underline{F}}\,\underline{i}}_{\underline{i}}+2\underline{\underline{F}}\,\underline{k}\implies \underline{\underline{F}}\,\underline{k}=\frac{1}{2}\underbrace{\underline{\underline{F}}\,\underline{t}}_{\underline{k}}-\frac{1}{2}\underline{i}=\frac{1}{2}\underline{k}-\frac{1}{2}\underline{i},$$

tehát
$$[\underline{\underline{F}}]^{\mathcal{K}}=\begin{bmatrix}1 & 0 & -1/2\\ 0 & 1 & 0\\ 0 & 0 & 1/2\end{bmatrix}$$

A két mátrix megegyezik, ahogy annak lennie kell!

c,
$$F^{-1}=\frac{1}{1/2}\cdot\begin{bmatrix}1/2 & 0 & 1/2\\ 0 & 1/2 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix},\ \text{és}$$

$$[\underline{\underline{A}}]^{\mathcal{K}}=F^{-1}[\underline{\underline{A}}]^{\mathcal{B}}F=\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}\underbrace{\begin{bmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -1/2\\ 0 & 1 & 0\\ 0 & 0 & 1/2\end{bmatrix}}_{\begin{bmatrix}\cos\alpha & -\sin\alpha & -(\cos\alpha)/2\\ \sin\alpha & \cos\alpha & -(\sin\alpha)/2\\ 0 & 0 & 1/2\end{bmatrix}}=$$

$$=\begin{bmatrix}\cos\alpha & -\sin\alpha & \frac{1-\cos\alpha}{2}\\ \sin\alpha & \cos\alpha & -\frac{\sin\alpha}{2}\\ 0 & 0 & 1\end{bmatrix}$$

$[\underline{\underline{A}}]^{\mathcal{K}}$ az F ismerete nélkül, közvetlenül is megkapható. Már láttuk, hogy $\underline{\underline{A}}\,\underline{i}=\underline{i}\cos\alpha+\underline{j}\sin\alpha$; $\underline{\underline{A}}\,\underline{j}=-\underline{i}\sin\alpha+\underline{j}\cos\alpha$; ez adja az első két oszlopot. $\underline{\underline{A}}\,\underline{t}=\underline{t}$, $\underline{t}=\underline{i}+2\underline{k}$, tehát

$$\underline{\underline{A}}(\underline{i}+2\underline{k})=\underbrace{\underline{\underline{A}}\,\underline{i}}_{\underline{i}\cos\alpha+\underline{j}\sin\alpha}+2\underline{\underline{A}}\,\underline{k}=\underline{i}+2\underline{k}\implies \underline{\underline{A}}\,\underline{k}=\frac{1-\cos\alpha}{2}\underline{i}-\frac{\sin\alpha}{2}\underline{j}+\underline{k}$$

Ez adja a 3. oszlopot.

4,

$$|A-\lambda I| = \begin{vmatrix} \frac{11}{5}-\lambda & -\frac{2i}{5} \\ \frac{2i}{5} & \frac{14}{5}-\lambda \end{vmatrix} = \lambda^2 - \lambda\left(\frac{11}{5}+\frac{14}{5}\right) + \frac{11\cdot 14}{25} - \frac{4}{25} =$$

$$= \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3); \quad \underline{\underline{\lambda_1 = 2}} \quad \underline{\underline{\lambda_2 = 3}}$$

$\boxed{\lambda_1 = 2}$

$$5(A-2I) = \begin{bmatrix} 1 & -2i \\ 2i & 4 \end{bmatrix}; \quad \begin{matrix} x - 2iy = 0 \\ 2ix + 4y = 0 \end{matrix} \Rightarrow \underline{\underline{\underline{v}_1 = \begin{bmatrix} 2 \\ -i \end{bmatrix} \frac{1}{\sqrt{5}}}}$$ sajátvektor

$$P_1 = \underline{v}_1 \cdot \underline{v}_1^* = \frac{1}{5}\begin{bmatrix} 2 \\ -i \end{bmatrix}\begin{bmatrix} 2 & i \end{bmatrix} = \underline{\underline{\frac{1}{5}\begin{bmatrix} 4 & 2i \\ -2i & 1 \end{bmatrix}}}$$ spektrál proj.

$\boxed{\lambda_2 = 3}$ A önadjungált, így normális, tehát $\underline{v}_2 \perp \underline{v}_1$, így $\underline{\underline{\underline{v}_2 = \begin{bmatrix} -i \\ 2 \end{bmatrix} \frac{1}{\sqrt{5}}}}$

$$P_1 + P_2 = I \Rightarrow P_2 = I - P_1 = \underline{\underline{\begin{bmatrix} 1 & -2i \\ 2i & 4 \end{bmatrix} \frac{1}{5}}}$$ Vagy közvetlen számolással:

$$5(A-3I) = \begin{bmatrix} -4 & -2i \\ 2i & -1 \end{bmatrix}; \quad \begin{matrix} -4x - 2iy = 0 \\ 2ix - y = 0 \end{matrix} \Rightarrow \underline{\underline{\underline{v}_2 = \begin{bmatrix} -i \\ 2 \end{bmatrix} \frac{1}{\sqrt{5}}}} \checkmark$$

$$P_2 = \underline{v}_2 \cdot \underline{v}_2^* = \frac{1}{5}\begin{bmatrix} -i \\ 2 \end{bmatrix}\begin{bmatrix} i & 2 \end{bmatrix} = \underline{\underline{\frac{1}{5}\begin{bmatrix} 1 & -2i \\ 2i & 4 \end{bmatrix}}} \checkmark$$

Spektrál felbontás:

$$\underline{\underline{A = \lambda_1 P_1 + \lambda_2 P_2 = 2\cdot\frac{1}{5}\begin{bmatrix} 4 & 2i \\ -2i & 1 \end{bmatrix} + 3\cdot\frac{1}{5}\begin{bmatrix} 1 & -2i \\ 2i & 4 \end{bmatrix}}} \checkmark$$

—∘—

5, $x(t) = (a - vt)\cos\left(\ln\left(\frac{a-vt}{a}\right)\right); \quad x_0 = x(t_0) = \frac{a}{2}\cos(\ln 1/2)$

$$\dot{x}(t) = -v\cos\left(\ln\left(\frac{a-vt}{a}\right)\right) - \cancel{(a-vt)}\sin\left(\ln\frac{a-vt}{a}\right)\cdot\frac{\cancel{a}}{\cancel{a-vt}}\cdot\left(\frac{-v}{\cancel{a}}\right) =$$

$$= -v\cos\left(\ln\left(\frac{a-vt}{a}\right)\right) + v\sin\left(\ln\left(\frac{a-vt}{a}\right)\right); \quad \dot{x}(t_0) = -v\cos(\ln 1/2) + v\sin(\ln 1/2)$$

$y(t) = (a-vt)\sin\left(\ln\left(\frac{a-vt}{a}\right)\right); \quad y_0 = y(t_0) = \frac{a}{2}\sin(\ln 1/2)$

$$\dot{y}(t) = -v\sin\left(\ln\left(\frac{a-vt}{a}\right)\right) + \cancel{(a-vt)}\cos\left(\ln\left(\frac{a-vt}{a}\right)\right)\cdot\frac{\cancel{a}}{\cancel{a-vt}}\cdot\left(\frac{-v}{\cancel{a}}\right) =$$

$$= -v\sin\left(\ln\left(\frac{a-vt}{a}\right)\right) - v\cos\left(\ln\left(\frac{a-vt}{a}\right)\right); \quad \dot{y}(t_0) = -v\sin(\ln 1/2) - v\cos(\ln 1/2)$$

Ívhossz:

$$S=\int_{t=0}^{a/v}\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}\,dt=\int_{t=0}^{a/v}\sqrt{2v^2\cos^2\left(\ln\left(\frac{a-vt}{a}\right)\right)+2v^2\sin^2\left(\ln\left(\frac{a-vt}{a}\right)\right)}\,dt=$$

(a kitevős mennyiségek kiesnek)

$$=\int_{t=0}^{a/v}\sqrt{2}\,v\,dt=\underline{\underline{\sqrt{2}\,a}}$$

Érintő: $(\ln 1/2=-\ln 2)$

$$\underline{t}=\begin{bmatrix}\dot{x}(t_0)\\ \dot{y}(t_0)\end{bmatrix}=\begin{bmatrix}-v\cos(\ln 2)-v\sin(\ln 2)\\ v\sin(\ln 2)-v\cos(\ln 2)\end{bmatrix}$$ irányvektor

normálvektor:

$$\underline{n}=v\begin{bmatrix}\sin(\ln 2)-\cos(\ln 2)\\ \sin(\ln 2)+\cos(\ln 2)\end{bmatrix}\sim\begin{bmatrix}\sin^2(\ln 2)-\cos^2(\ln 2)\\ 1+2\sin(\ln 2)\cos(\ln 2)\end{bmatrix}=$$

$$=\begin{bmatrix}-\cos(2\ln 2)\\ 1+\sin(2\ln 2)\end{bmatrix}=\begin{bmatrix}-\cos(\ln 4)\\ 1+\sin(\ln 4)\end{bmatrix}$$

Érintő általában: $n_x\cdot(x-x_0)+n_y(y-y_0)=0$

Most:

$$\underline{\underline{-\cos(\ln 4)\left(x-\frac{a}{2}\cos(\ln 2)\right)+(1+\sin(\ln 4))\left(y+\frac{a}{2}\sin(\ln 2)\right)=0}}$$