1, a,

[3]
$$\det\begin{bmatrix} a_{11} & - & a_{1n} \\ | & & \\ a_{n1} & - & a_{nn} \end{bmatrix} = \sum_{\sigma \in S_n} I(\sigma)\cdot a_{1,\sigma(1)} \cdot a_{2,\sigma(2)} \cdots a_{n,\sigma(n)},$$

ahol S_n az n-edrendű permutáció-csoport, tehát $\sigma \in S_n$ az $\{1 .. n\}$ halmaz egy permutációja, és $I(\sigma) = \begin{cases} +1, & \text{ha } \sigma \text{ páros} \\ -1, & \text{ha } \sigma \text{ páratlan} \end{cases}$ permutáció.

[2] b, <u>T.:</u> Ha két sor egyezik, azaz pl.: $a_{i,k} = a_{j,k}$ $\forall k \in \{1 .. n\}$, ahol $i, j \in \{1 .. n\}$, $i \neq j$, akkor a determináns zérus.

c, <u>B.:</u> Ha $a_{i,k} = a_{j,k}$ $\forall k \in \{1 .. n\}$, akkor *

[5]
$$a_{1,\sigma(1)} \cdots a_{i,\sigma(i)} \cdots a_{j,\sigma(j)} \cdots a_{n,\sigma(n)} = \prod_l a_{l,\sigma(l)} =$$

$$= a_{1,\sigma(1)} \cdots \underset{\substack{\uparrow \\ a_{i,\sigma'(i)}}}{a_{i,\sigma(j)}} \cdots \underset{\substack{\uparrow \\ a_{j,\sigma'(j)}}}{a_{j,\sigma(i)}} \cdots a_{n,\sigma(n)} = \prod_l a_{l,\sigma'(l)}$$

* Legyen
$$\sigma'(l) = \begin{cases} \sigma(l), & \text{ha } l \neq i,\ l \neq j \\ \sigma(j), & \text{ha } l = i \\ \sigma(i), & \text{ha } l = j \end{cases}$$

Továbbá $I(\sigma) = -I(\sigma')$. Ez azt jelenti, hogy a determinánsban szereplő tagok páronként kiejtik egymást.

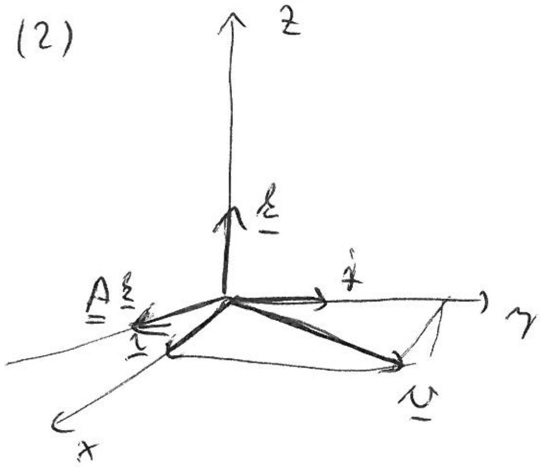
2, Legyen $\underline{i}=\begin{bmatrix}1\\0\\0\end{bmatrix}$; $\underline{j}=\begin{bmatrix}0\\1\\0\end{bmatrix}$; $\underline{k}=\begin{bmatrix}0\\0\\1\end{bmatrix}$; Ekkor $\underline{v}=\underline{i}+2\underline{j}$

10

$$\underline{\underline{A}}\,\underline{v}=2\underline{v} \implies \underline{\underline{A}}\,\underline{i}+2\underline{\underline{A}}\,\underline{j}=2\underline{i}+4\underline{j} \qquad (1)$$

Mivel $\underline{k}\perp\underline{v}$, ezért $\underline{\underline{A}}\,\underline{k}$ az $(\underline{i},\underline{j})$ síkba esik, és $\underline{\underline{A}}\,\underline{k}\perp\underline{v}$,

tehát $$\underline{\underline{A}}\,\underline{k}=\frac{1}{\sqrt5}\begin{bmatrix}2\\-1\\0\end{bmatrix}=\frac{2}{\sqrt5}\underline{i}-\frac{1}{\sqrt5}\underline{j} \qquad (2)$$



$\underline{\underline{A}}^2\underline{k}=-\underline{k}$, tehát

$$\frac{2}{\sqrt5}\underline{\underline{A}}\,\underline{i}-\frac{1}{\sqrt5}\underline{\underline{A}}\,\underline{j}=-\underline{k} \qquad (3)$$

$(3)\cdot2\sqrt5+(1)$: $5\underline{\underline{A}}\,\underline{i}=2\underline{i}+4\underline{j}-2\sqrt5\,\underline{k}$

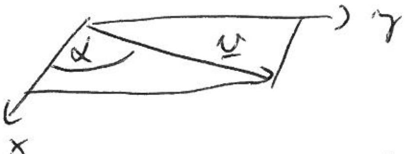
$(3)-(1)\cdot\frac{2}{\sqrt5}$: $-\sqrt5\,\underline{\underline{A}}\,\underline{j}=-\frac{4}{\sqrt5}\underline{i}-\frac{8}{\sqrt5}\underline{j}-\underline{k}$

Tehát $$[\underline{\underline{A}}]=\left[\,[\underline{\underline{A}}\,\underline{i}]\;\middle|\;[\underline{\underline{A}}\,\underline{j}]\;\middle|\;[\underline{\underline{A}}\,\underline{k}]\,\right]=\begin{bmatrix}2/5 & 4/5 & 2/\sqrt5\\ 4/5 & 8/5 & -1/\sqrt5\\ -2/\sqrt5 & 1/\sqrt5 & 0\end{bmatrix}$$

10 Másképp. Legyen $\underline{\underline{O}}_z^{\alpha}$ a z körüli α szögű forgatás, $\underline{\underline{O}}_x^{\beta}$ pedig az x tengely körüli β szögű, és $N_x^{(2)}$ az x irányú 2-szeres nyújtás

$\underline{\underline{A}}=\underline{\underline{O}}_z^{\alpha}N_x^{(2)}\underline{\underline{O}}_x^{90°}\underline{\underline{O}}_z^{-\alpha}$, ahol α az x tengely és v szöge, tehát

$$\cos\alpha=\frac{1}{\sqrt5};\quad \sin\alpha=\frac{2}{\sqrt5}$$



$$A=\underbrace{\begin{bmatrix}\frac{1}{\sqrt5} & -2/\sqrt5 & 0\\ 2/\sqrt5 & 1/\sqrt5 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}}_{\begin{bmatrix}2/\sqrt5 & 0 & 2/\sqrt5\\ 4/\sqrt5 & 0 & -1/\sqrt5\\ 0 & 1 & 0\end{bmatrix}}\begin{bmatrix}1/\sqrt5 & 2/\sqrt5 & 0\\ -2/\sqrt5 & 1/\sqrt5 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}2/5 & 4/5 & 2/\sqrt5\\ 4/5 & 8/5 & -1/\sqrt5\\ -2/\sqrt5 & 1/\sqrt5 & 0\end{bmatrix}$$

3, $\boxed{10}$

$$\left[\begin{array}{cccc|c} 2 & -1 & -2 & 1 & 0 \\ -1 & +1 & 2 & 2 & -7 \\ 3 & +2 & +4 & 0 & 8 \\ 1 & 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} (-2) \\ +1 \\ (-3) \end{array} \Rightarrow \left[\begin{array}{cccc|c} 0 & -1 & -2 & -1 & 2 \\ 0 & 1 & 2 & 3 & -8 \\ 0 & 2 & 4 & -3 & 11 \\ 1 & 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} +1 \\ (-2)\times \end{array} \Rightarrow$$

$$\Rightarrow \left[\begin{array}{cccc|c} 0 & 0 & 0 & 2 & -6 \\ 0 & 1 & 2 & 3 & -8 \\ 0 & 0 & 0 & -9 & 27 \\ 1 & 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{c} /:2 \\ \\ /:-9 \end{array} \Rightarrow \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & -3 \\ 0 & 1 & 2 & 3 & -8 \\ 0 & 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & -3 \\ 0 & 1 & 2 & 0 & +1 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 1 & 0 & 0 & 0 & +2 \end{array}\right]$$

Tehát $x = 2,\ y + 2z = 1;\ w = -3$ (8); A rang $= 3$ (2) (vezéregyesek száma)

4, a, $\boxed{6}$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 2 = \lambda^2 - 2\lambda - 1$$

$$\lambda_{1,2} = 1 \pm \sqrt{1+1} = 1 \pm \sqrt{2} \quad (2)$$

$$\lambda_1 = 1+\sqrt{2}\,; \quad \underbrace{\begin{bmatrix} -\sqrt{2} & 2 \\ 1 & -\sqrt{2} \end{bmatrix}}_{A-\lambda_1 I} \begin{bmatrix} x \\ y \end{bmatrix} = \underline{0} \Rightarrow \begin{array}{c} -\sqrt{2}x + 2y = 0 \\ x - \sqrt{2}y = 0 \end{array} \Rightarrow \underline{v}_1 = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix} \quad (2)$$

$$\lambda_2 = 1-\sqrt{2}\,; \quad \begin{bmatrix} \sqrt{2} & 2 \\ 1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \underline{0} \Rightarrow \begin{array}{c} \sqrt{2}x + 2y = 0 \\ x + \sqrt{2}y = 0 \end{array} \Rightarrow \underline{v}_2 = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix} \quad (2)$$

b, $\boxed{4}$

$$B(p)\cdot B^*(p) - B^*(p)\,B(p) = \begin{bmatrix} 1 & p \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ p & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ p & 1 \end{bmatrix}\begin{bmatrix} 1 & p \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+p^2 & 1+p \\ 1+p & 2 \end{bmatrix} -$$

$$- \begin{bmatrix} 2 & 1+p \\ 1+p & 1+p^2 \end{bmatrix} = \begin{bmatrix} p^2-1 & 0 \\ 0 & 1-p^2 \end{bmatrix} \Rightarrow$$ B normális, ha $p^2 - 1 = 0$, azaz ha $p = \pm 1$ (2)

$B(p)$ önadjungált, ha $p = 1$ (1)

$B(p)$ invertálható, ha $\det B(p) = 1 - p \neq 0$, azaz $p \neq 1$ (1)

5, a, $\boxed{6}$ $\operatorname{div}(\underline{r} \times \underline{v}(\underline{r})) = \partial_i(\varepsilon_{jki}\, r_j\, v_k) = \varepsilon_{jki}\left(\overbrace{(\partial_i r_j)}^{\delta_{ij}} v_k + r_j(\partial_i v_k)\right) =$

$$= \underbrace{\varepsilon_{iki}}_{0} v_k + r_j\, \varepsilon_{jki}(\partial_i v_k) = \underline{\underline{-\underline{r} \cdot \operatorname{rot} \underline{v}}}$$

b, $\boxed{4}$ $\underline{r} \times \underline{v}(\underline{r}) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \times \begin{bmatrix} x^2 y \\ 2x - yz \\ 3z \end{bmatrix} = \begin{bmatrix} 3yz - 2xz + yz^2 \\ x^2yz - 3xz \\ 2x^2 - xyz - x^2y^2 \end{bmatrix};$

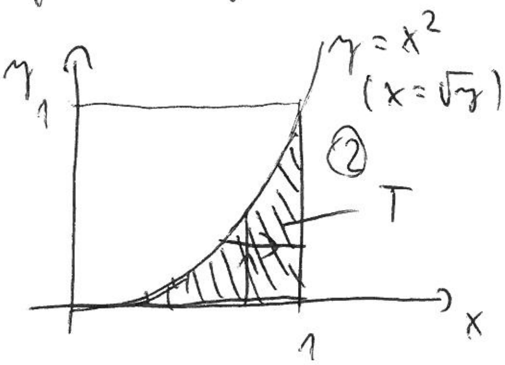
$$\operatorname{div}(\underline{r} \times \underline{v}(\underline{r})) = \underline{\underline{-2z + zx^2 - xy}}$$

Vagy másképp: $\boxed{4}$ $\operatorname{rot} \underline{v} = \begin{bmatrix} 0 - (-y) \\ 0 - 0 \\ 2 - x^2 \end{bmatrix} = \begin{bmatrix} y \\ 0 \\ 2 - x^2 \end{bmatrix};$

$$-\underline{r} \cdot \operatorname{rot} \underline{v} = \underline{\underline{-xy - 2z + zx^2}} \quad ✓$$

— o —

6, $\boxed{10}$ $\displaystyle\int_{y=0}^{1} \int_{x=\sqrt{y}}^{1} \sin(x^3)\, dx\, dy = \int_{x=0}^{1} \int_{y=0}^{x^2} \sin(x^3)\, dy\, dx =$ ④



$$= \int_{x=0}^{1} \sin(x^3) \cdot x^2\, dx = \frac{1}{3} \int_{x=0}^{1} \sin(x^3)(3x^2)\, dx =$$

$\sin \varphi \cdot \varphi'$ alakú

$$= \left[\frac{-\cos(x^3)}{3}\right]_0^1 = \underline{\underline{\frac{-\cos(1)}{3} + \frac{1}{3}}}$$ ④