Szám. Módsz. Fiz. 1. 3. vizsga (2015.01.15.) |-1-|

MINTAMEGOLDÁS

1, a, D.: Az $A:\mathbb{V}\to\mathbb{V}$ lin. tr. adjungáltja az $A^*:\mathbb{V}\to\mathbb{V}$ lin. tr., ha

[3] $\forall\, x,y\in\mathbb{V}$ esetén $\langle x, Ay\rangle = \langle A^*x, y\rangle$. (1)

A unitér, ha $A^{-1}=A^*$ (1); A normális, ha $A^*A=AA^*$. (1)

[2] b, T.: Unitér tr. sajátértékei egységnyi abszolútértékűek. (2)

[5] c, B.: Legyen U unitér ($U^*=U^{-1}$) és $Uv=\lambda v$; $v\neq 0$.

Ekkor $\|\lambda v\|^2=\langle\lambda v,\lambda v\rangle = |\lambda|^2\|v\|^2 = \langle Uv, Uv\rangle = \langle v, U^*Uv\rangle =$

$= \langle v, \underbrace{U^{-1}U}_{I} v\rangle = \|v\|^2 \Longrightarrow |\lambda|^2=1 \Longrightarrow |\lambda|=1$ ✓ (5)

2, $\vec{A}=\begin{bmatrix}1\\1\\-1\end{bmatrix}$; $\vec{B}=\begin{bmatrix}-1\\1\\1\end{bmatrix}$; $\vec{C}=\begin{bmatrix}1\\-1\\1\end{bmatrix}$; $\vec{D}=\begin{bmatrix}-1\\-1\\-1\end{bmatrix}$

$$\underline{i}=\begin{bmatrix}1\\0\\0\end{bmatrix}=\frac{\vec{A}+\vec{C}}{2};\quad \phi(\underline{i})=\frac{1}{2}\phi(\vec{A})+\frac{1}{2}\phi(\vec{C})=\frac{\vec{B}+\vec{D}}{2}=\begin{bmatrix}-1\\0\\0\end{bmatrix}$$

$$\underline{j}=\begin{bmatrix}0\\1\\0\end{bmatrix}=\frac{\vec{A}+\vec{B}}{2};\quad \phi(\underline{j})=\frac{1}{2}\phi(\vec{A})+\frac{1}{2}\phi(\vec{B})=\frac{\vec{B}+\vec{C}}{2}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

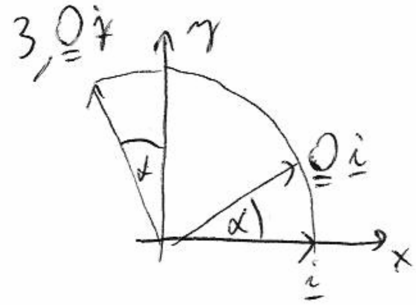
$$\underline{k}=\begin{bmatrix}0\\0\\1\end{bmatrix}=\frac{\vec{B}+\vec{C}}{2};\quad \phi(\underline{k})=\frac{1}{2}\phi(\vec{B})+\frac{1}{2}\phi(\vec{C})=\frac{\vec{C}+\vec{D}}{2}=\begin{bmatrix}0\\-1\\0\end{bmatrix}$$

Tehát $[\phi]=\begin{bmatrix}-1&0&0\\0&0&-1\\0&1&0\end{bmatrix}$ (6); $\phi(\vec{D})=\begin{bmatrix}-1&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}-1\\-1\\-1\end{bmatrix}=\begin{bmatrix}1\\1\\-1\end{bmatrix}$ (2)

Térfogat növelés $=|\phi|=\begin{vmatrix}-1&0&0\\0&0&-1\\0&1&0\end{vmatrix}=-1\underbrace{\begin{vmatrix}0&-1\\1&0\end{vmatrix}}_{+1}=-1\cdot 1=-1$ (2)

Tehát a térfogat nem változott, az orientáció megfordult.

3,



$$O(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \quad ②$$

Legyen K a kísérő mátrix. $K\underline{i} = \underline{f}$; $K\underline{j} = \underline{g}$, tehát

$$K = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix} ② \text{ (akár az } (\underline{i}, \underline{j})\text{, akár az } (\underline{f}, \underline{g}) \text{ bázisban)}$$

$$K^{-1} = \frac{1}{-1} \cdot \begin{bmatrix} -1 & +1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \quad ②$$

$$\tilde{O}(\alpha) = K^{-1} O(\alpha) K \overset{②}{=} \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}}_{\begin{bmatrix} 2\cos\alpha - \sin\alpha & -\cos\alpha + \sin\alpha \\ 2\sin\alpha + \cos\alpha & -\sin\alpha - \cos\alpha \end{bmatrix}} =$$

$$= \begin{bmatrix} -3\sin\alpha + \cos\alpha & 2\sin\alpha \\ -5\sin\alpha & 3\sin\alpha + \cos\alpha \end{bmatrix} \quad ②$$

4, $\det(A - \lambda I) = \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \implies \lambda_{1,2} = \pm 1$ ②

$\lambda_1 = +1$; $\begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{matrix} -x - iy = 0 \\ ix - y = 0 \end{matrix} \implies \underline{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$ ②

$$P_1 = \underline{v}_1 \circ \underline{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} [1; -i] \cdot \frac{1}{2} = \frac{1}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

$\lambda_2 = -1$; $\begin{bmatrix} +1 & -i \\ i & +1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{matrix} x - iy = 0 \\ ix + y = 0 \end{matrix} \implies \underline{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix} \cdot \frac{1}{\sqrt{2}}$ ②

$$P_2 = \underline{v}_2 \circ \underline{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix} [1; i] \cdot \frac{1}{2} = \frac{1}{2} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = I - P_1 \checkmark$$

Spektrál felbontás: $A = \lambda_1 P_1 + \lambda_2 P_2 = +\frac{1}{2} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$ ✓ ②

$\frac{1}{A} = A^{-1} = \frac{1}{\underbrace{\lambda_1}_{1 = \lambda_1}} P_1 + \frac{1}{\underbrace{\lambda_2}_{-1 = \lambda_2}} P_2 = A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ ② (Látható, hogy $A^2 = I$)

5, $f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & \text{ha } (x,y)\neq(0,0)\\ 0, & \text{ha } (x,y)=(0,0)\end{cases}$

a, $\boxed{4}$

Ha $(x,y)\neq(0,0)$, akkor

$$\operatorname{grad} f(x,y)=\begin{bmatrix} f'_x(x,y)\\ f'_y(x,y)\end{bmatrix}=\begin{bmatrix} \dfrac{y\sqrt{x^2+y^2}-xy\dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}}{x^2+y^2}\\ f'_y(x,y)\end{bmatrix}=\begin{bmatrix} \dfrac{y(\cancel{x^2}+y^2)-\cancel{yx^2}}{(x^2+y^2)^{3/2}}\\ f'_y(x,y)\end{bmatrix}=\begin{bmatrix} \dfrac{y^3}{(x^2+y^2)^{3/2}}\\ \dfrac{x^3}{(x^2+y^2)^{3/2}}\end{bmatrix}$$

$x\leftrightarrow y$ ③

$$\operatorname{grad} f(2,-1)=\begin{bmatrix} \dfrac{-1}{5^{3/2}}\\ \dfrac{8}{5^{3/2}}\end{bmatrix}=\underline{\underline{\frac{1}{5\cdot\sqrt5}\begin{bmatrix}-1\\ 8\end{bmatrix}}}$$ ①

($\exists \operatorname{grad} f(2,-1)$, mert $(2,-1)$ egy környezetében f'_x és f'_y folytonos.)

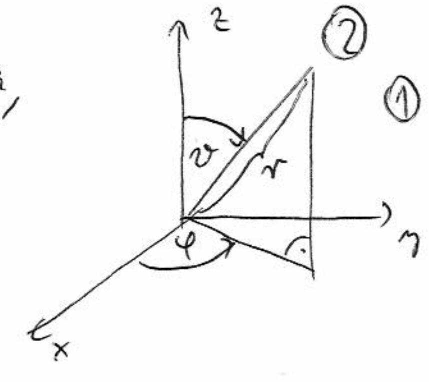
b, $\boxed{3}$

$$\left.\frac{df}{d\underline{e}}\right|_A=\langle \underline{e}, \operatorname{grad} f(A)\rangle \text{ ②} =\frac15\begin{bmatrix}3\\4\end{bmatrix}\cdot\frac{1}{5\sqrt5}\begin{bmatrix}-1\\8\end{bmatrix}=\frac{-3+32}{25\sqrt5}=\underline{\underline{\frac{29}{25\sqrt5}}}$$ ①

Az origóban a definícióval számolunk:

$\boxed{3}$ $$\left.\frac{df}{d\underline{e}}\right|_{\underline{0}}=\lim_{t\to0+}\frac{f(t\underline{e})-f(\underline{0})}{t} \text{ ②} =\lim_{t\to0+}\frac1t\cdot\left(\frac{t^2\cdot\frac35\cdot\frac45}{t}-0\right)=\underline{\underline{\frac{12}{25}}}$$ ①

6, a, $\boxed{5}$



②

① $\begin{cases} x=r\sin\vartheta\cos\varphi\\ y=r\sin\vartheta\sin\varphi\\ z=r\cos\vartheta\end{cases}$; $J(r,\vartheta,\varphi)=\begin{vmatrix} x'_r & x'_\vartheta & x'_\varphi\\ y'_r & y'_\vartheta & y'_\varphi\\ z'_r & z'_\vartheta & z'_\varphi\end{vmatrix}=$

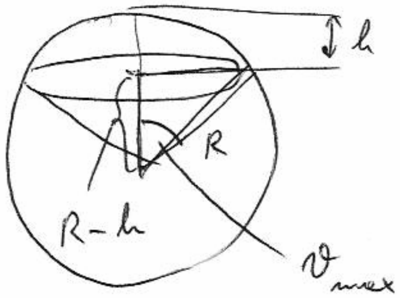
$$=\begin{vmatrix} \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi\\ \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi\\ \cos\vartheta & -r\sin\vartheta & 0\end{vmatrix}=$$

$$=\cos\vartheta\underbrace{\begin{vmatrix} r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi\\ r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi\end{vmatrix}}_{r^2\sin\vartheta\cos\vartheta}-(-r\sin\vartheta)\underbrace{\begin{vmatrix} \sin\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi\\ \sin\vartheta\sin\varphi & r\sin\vartheta\cos\varphi\end{vmatrix}}_{r\sin^2\vartheta}=$$

$$=r^2\sin\vartheta\cos^2\vartheta+r^2\sin^3\vartheta=\underline{\underline{r^2\sin\vartheta}}$$ ②

b,

$\boxed{5}$



$$V: \left.\begin{array}{l} 0 \le r \le R \\ 0 \le \vartheta \le \vartheta_{max} \\ 0 \le \varphi \le 2\pi \end{array}\right\} \ (1)$$

(2)

$$V = \int_{r=0}^{R} \int_{\vartheta=0}^{\vartheta_{max}} \int_{\varphi=0}^{2\pi} 1 \cdot \underbrace{r^2 \sin\vartheta}_{J} \, d\varphi \, d\vartheta \, dr =$$

$$= \left(\int_{r=0}^{R} r^2 \, dr\right) \cdot \left(\int_{\vartheta=0}^{\vartheta_{max}} \sin\vartheta \, d\vartheta\right) \cdot \left(\int_{\varphi=0}^{2\pi} d\varphi\right) = \frac{R^3}{3} \cdot \underbrace{\Big(1 - \underbrace{\cos\vartheta_{max}}_{\frac{R-h}{R}}\Big)}_{\frac{h}{R}} \cdot 2\pi =$$

$$= \underline{\underline{\frac{2\pi R^2 h}{3}}} \ (2)$$