1, a, **D.:** A V vektortérben az $\{\underline{v}_i\}_{i\in I} \subset V$ rendszer <u>lineárisan független</u>, ha $\sum_{k=1}^{n} \alpha_k \underline{v}_{i_k} = \underline{0}$ esetén $\alpha_k = 0$ $\forall k \in \{1,2,\dots n\}$ $(i_k \in I)$

D.: $\{\underline{v}_i\}_{i\in I}$ <u>generátorrendszer</u>, ha $\forall \underline{x} \in V$ esetén $\exists\, i_k \in I$, $\alpha_k \in \mathbb{R}$ $(k=1,2,\dots n)$, hogy $\sum_{k=1}^{n} \alpha_k \underline{v}_{i_k} = \underline{x}$

D.: $\{\underline{v}_i\}_{i\in I} \subset V$ <u>bázis V-ben</u>, ha lin. független és generátorrendszer.

b, Legyen $\{\underline{b}_i\}_{i\in I} \subset V$ bázis. Mivel generátorrendszer, ezért minden vektor kifejthető. T.f.h. $\underline{v} \in V$ kétféle képpen is kifejthető, azaz

$$\underline{v} = \sum_{k=1}^{n} \alpha_k \underline{b}_{i_k} = \sum_{l=1}^{m} \beta_l \underline{b}_{j_l}.$$

Ekkor

$$\underline{0} = \sum_{k=1}^{n} \alpha_k \underline{b}_{i_k} - \sum_{l=1}^{m} \beta_l \underline{b}_{j_l}, \quad \alpha_k, \beta_l \neq 0,$$

ami ellentmond annak, hogy $\{\underline{b}_i\}_{i\in I}$ lin. független rendszer.

2, a, $p_1, p_2 \in P$; $\alpha, \beta \in \mathbb{R}$ esetén

$$(A(\alpha p_1 + \beta p_2))(x) = (\alpha p_1 + \beta p_2)(x-1) = \alpha p_1(x-1) + \beta p_2(x-1) =$$

$$= \alpha (A p_1)(x) + \beta (A p_2)(x) \quad \forall x \in \mathbb{R} \text{ esetén, tehát}$$

$$A(\alpha p_1 + \beta p_2) = \alpha A p_1 + \beta A p_2 \quad ✓$$

b, Legyen $b_k(x) = x^k$, $k = 0,1,2,3$.

$$(A b_0)x = b_0(x-1) = 1$$

$$(A b_1)(x) = b_1(x-1) = x-1 = (b_1 - b_0)(x)$$

$$(A b_2)(x) = b_2(x-1) = (x-1)^2 = x^2 - 2x + 1 =$$

$$= (b_2 - 2b_1 + b_0)(x)$$

$$(A b_3)(x) = b_3(x-1) = (x-1)^3 = x^3 - 3x^2 + 3x - 1 =$$

$$= (b_3 - 3b_2 + 3b_1 - b_0)(x)$$

$$[A] = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\lfloor -2 - \rceil$

3, $A=\begin{bmatrix}2 & -2\\ -2 & -1\end{bmatrix}$; $|A-\lambda I| = \begin{vmatrix}2-\lambda & -2\\ -2 & -1-\lambda\end{vmatrix} = \lambda^2-\lambda-2-4 = \lambda^2-\lambda-6 =$

$= (\lambda-3)(\lambda+2) \Rightarrow \lambda_1=-2;\ \lambda_2=3$

$\lambda_1=-2$: $\underbrace{\begin{bmatrix}4 & -2\\ -2 & 1\end{bmatrix}}_{A-\lambda_1 I}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \Rightarrow \begin{array}{l}4x-2y=0\\ -2x+y=0\end{array} \Rightarrow \underline{v}_1=\begin{bmatrix}1\\ 2\end{bmatrix}\cdot\frac{1}{\sqrt5}$

$P_1=\underline{v}_1\circ\underline{v}_1=\frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$

$\lambda_2=+3$ $\quad P_2=I-P_1=\frac{1}{5}\begin{bmatrix}4 & -2\\ -2 & 1\end{bmatrix}$

$\underbrace{\begin{bmatrix}-1 & -2\\ -2 & -4\end{bmatrix}}_{A-\lambda_2 I}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \Rightarrow \begin{array}{l}-x-2y=0\\ -2x-4y=0\end{array} \Rightarrow \underline{v}_2=\begin{bmatrix}2\\ -1\end{bmatrix}\frac{1}{\sqrt5}$

A spektrál felbontása: $A=\lambda_1 P_1+\lambda_2 P_2=-2\cdot\frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}+3\cdot\frac{1}{5}\begin{bmatrix}4 & -2\\ -2 & 1\end{bmatrix}$

$e^A=e^{\lambda_1}P_1+e^{\lambda_2}P_2=e^{-2}\cdot\frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}+e^{3}\frac{1}{5}\begin{bmatrix}4 & -2\\ -2 & 1\end{bmatrix}$

4, $\underline{r}(t)=\begin{bmatrix}t\\ t^2\\ t^3\end{bmatrix}$; $\dot{\underline{r}}(t)=\begin{bmatrix}1\\ 2t\\ 3t^2\end{bmatrix}$; $\ddot{\underline{r}}(t)=\begin{bmatrix}0\\ 2\\ 6t\end{bmatrix}$

$\underline{r}(t_0)=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$; $\dot{\underline{r}}(t_0)=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$; $\ddot{\underline{r}}(t_0)=\begin{bmatrix}0\\ 2\\ 6\end{bmatrix}$; $\dot{\underline{r}}(t_0)\times\ddot{\underline{r}}(t_0)=\begin{bmatrix}6\\ -6\\ 2\end{bmatrix}$

$G(t_0)=\frac{|\dot{\underline{r}}(t_0)\times\ddot{\underline{r}}(t_0)|}{|\dot{\underline{r}}(t_0)|^3}=\frac{\sqrt{36+36+4}}{(\sqrt{1+4+9})^3}=\frac{\sqrt{76}}{\sqrt{14^3}}$

A simuló sík normálvektora arányos $\dot{\underline{r}}\times\ddot{\underline{r}}$-tal; $\underline{b}=\begin{bmatrix}3\\ -3\\ 1\end{bmatrix}$

A sík egyenlete:

$\boxed{3x-3y+z=1}$

$\underline{r}(t_0)\cdot\underline{n}=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}\cdot\begin{bmatrix}3\\ -3\\ 1\end{bmatrix}=3-3+1=1$

5.*

a, $[\mathrm{rot}(f\underline{v})]_i = \varepsilon_{ijk}\,\partial_j(f v_k) = \varepsilon_{ijk}\left((\partial_j f) v_k + f \partial_j v_k\right) =$

$= [\mathrm{grad}\, f \times \underline{v} + f\, \mathrm{rot}\, v]_i$

b, $f(x,y,z) = x^2 y + z$; $\mathrm{grad}\, f = \begin{bmatrix} 2xy \\ x^2 \\ 1 \end{bmatrix}$; $v = \begin{bmatrix} xy \\ z^2+2x \\ xyz \end{bmatrix}$

$$\mathrm{grad}\, f \times \underline{v} = \begin{bmatrix} x^3yz - (z^2+2x) \\ xy - 2x^2y^2z \\ 2xy(z^2+2x) - x^3y \end{bmatrix}$$

$$\mathrm{rot}\, \underline{v} = \begin{bmatrix} xz - 2z \\ 0 - yz \\ 2 - x \end{bmatrix}; \quad f\, \mathrm{rot}\, \underline{v} = \begin{bmatrix} (xz - 2z)(x^2y+z) \\ -yz(x^2y+z) \\ (2-x)(x^2y+z) \end{bmatrix}$$

jobb oldal: $\begin{bmatrix} x^3yz - (z^2+2x) + (xz-2z)(x^2y+z) \\ xy - 2x^2y^2z - yz(x^2y+z) \\ 2xy(z^2+2x) - x^3y + (2-x)(x^2y+z) \end{bmatrix}$

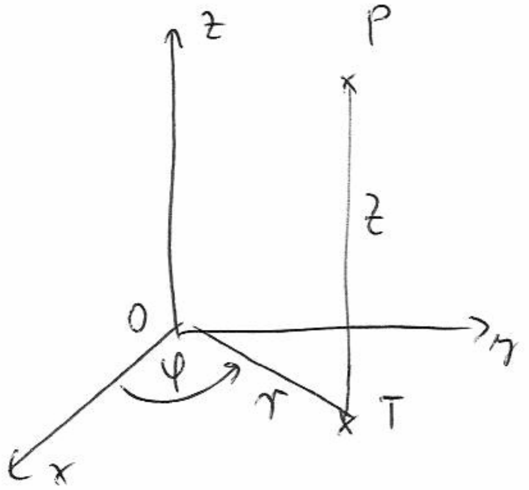
$$f\underline{v} = \begin{bmatrix} x^3y^2 + xyz \\ x^2yz^2 + z^3 + 2x^3y + 2xz \\ x^3y^2z + xyz^2 \end{bmatrix};$$

bal oldal:

$$\mathrm{rot}(f\underline{v}) = \begin{bmatrix} 2x^3yz + xz^2 - (2x^2yz + 3z^2 + 2x) \\ xy - 3x^2y^2z - yz^2 \\ 2xyz^2 + 6x^2y + 2z - 2x^3y - xz \end{bmatrix}$$

6*

a)



$r = \overline{OT}\,;\; z = \overline{TP}$

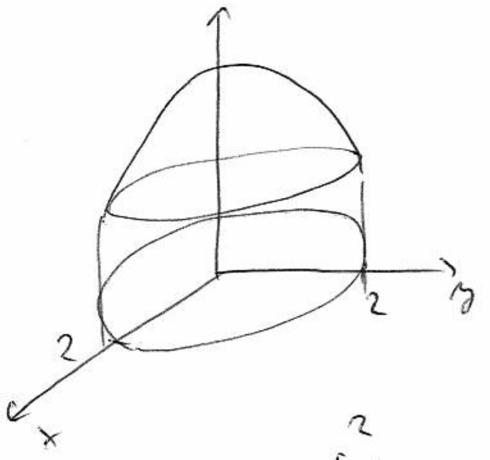
$$\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z \end{aligned}\right\}$$

$$J(r,\varphi,z) = \begin{vmatrix} x'_r & x'_\varphi & x'_z \\ y'_r & y'_\varphi & y'_z \\ z'_r & z'_\varphi & z'_z \end{vmatrix} =$$

$$= \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = r(\cos^2\varphi + \sin^2\varphi) = \underline{\underline{r}}$$

b)

$$\left.\begin{aligned} x^2+y^2 &\le 4 \\ 0 \le z &\le 6-x^2-y^2 \end{aligned}\right\} \iff \left.\begin{aligned} 0 &\le r \le 2 \\ 0 &\le \varphi \le 2\pi \\ 0 &\le z \le 6-r^2 \end{aligned}\right\}$$



$$V = \int_{\varphi=0}^{2\pi}\int_{r=0}^{2}\int_{z=0}^{6-r^2} 1\cdot \underbrace{r}_{\text{jac. det.}}\, dz\,dr\,d\varphi =$$

$$= 2\pi \int_{r=0}^{2} r\,(6-r^2-0)\,dr =$$

$$= 2\pi \int_{r=0}^{2} (6r-r^3)\,dr = 2\pi\left[-\frac{r^4}{4} + 6\,\frac{r^2}{2}\right]_0^2 =$$

$$= 2\pi(-4+3\cdot 4) = \underline{\underline{16\pi}}$$