Time dependent harmonic oscillator

Consider the following time dependent Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(x - d(t))^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 - m\omega^2 x d(t) + \frac{1}{2}m\omega^2 d^2(t) \ .$$

The Hamiltonian can be rewritten using the usual ladder operators:

$$H = \hbar\omega\left(a^+ a + \frac{1}{2}\right) - \hbar\omega\left(\frac{d(t)}{\sqrt{2}x_0}(a + a^+) + \frac{1}{2}\frac{d^2(t)}{x_0^2}\right)$$

In interaction picture the time dependent Schrödinger equation can be written as:

$$i\hbar\partial_t\psi(t) = -\hbar\omega\left(\frac{d(t)}{\sqrt{2}x_0}(ae^{-i\omega t} + a^+ e^{i\omega t}) - \frac{1}{2}\frac{d^2(t)}{x_0^2}\right)\psi(t)$$

The solution is supposed to have the following form:

$$\psi(t) = e^{i(\alpha(t)^* a^+ + \alpha(t) a + \beta(t))}\psi(0)$$

Using the Baker-Campbell-Hausdorff relation the exponential operator can be rewritten into a more appropriate form:

$$\psi(t) = e^{-\frac{|\alpha|^2}{2} + i\beta} e^{i\alpha^* a^+} e^{i\alpha a}\psi(0)$$

Calculate the partial derivative of the wavefunction with respect of the time:

$$\partial_t\psi = e^{-\frac{|\alpha|^2}{2} + i\beta}\left((\frac{1}{2}(\dot{\alpha}^*\alpha + \alpha^*\dot{\alpha}) + i\dot{\beta})e^{i\alpha^* a^+}e^{i\alpha a} + i\dot{\alpha}^* a^+ e^{i\alpha^* a^+}e^{i\alpha a} + e^{i\alpha^* a^+} i\dot{\alpha} a e^{i\alpha a}\right)\psi(0)$$

We should reverse the order of the ladder and the exponential operators in the last term:

$$e^{i\alpha^* a^+} i\dot{\alpha} a e^{i\alpha a} = i\dot{\alpha} a e^{i\alpha^* a^+} e^{i\alpha a} + i\dot{\alpha}[e^{i\alpha^* a^+}, a]e^{i\alpha a}$$

In order to evaluate the commutaion rule in the above expression we use the relation: $[a^{+n}, a] = -na^{+n-1}$.

$$[e^{i\alpha^* a^+}, a] = \sum_{n+0}^{\infty}\frac{(i\alpha^*)^n}{n!}[a^{+n}, a] = -\sum_{n+0}^{\infty}\frac{(i\alpha^*)^n}{n!}na^{+n-1} = -i\alpha^*\sum_{n+0}^{\infty}\frac{(i\alpha^*)^{n-1}}{(n-1)!}a^{+n-1} = -i\alpha^* e^{i\alpha^* a^+}$$

consequently

$$e^{i\alpha^* a^+} i\dot{\alpha} a e^{i\alpha a} = (i\dot{\alpha} a - i\dot{\alpha}\alpha^*)e^{i\alpha^* a^+}e^{i\alpha a}$$

Now one can obtain the partial derivatives of the wavefunction in the following form:

$$\begin{aligned}\partial_t\psi &= \left(\frac{1}{2}(\dot{\alpha}^*\alpha - \alpha^*\dot{\alpha}) + i\dot{\beta} + i\dot{\alpha}^* a^+ + i\dot{\alpha} a\right) e^{-\frac{|\alpha|^2}{2} + i\beta} e^{i\alpha^* a^+} e^{i\alpha a}\psi(0) \\ &= \left(\frac{1}{2}(\dot{\alpha}^*\alpha - \alpha^*\dot{\alpha}) + i\dot{\beta} + i\dot{\alpha}^* a^+ + i\dot{\alpha} a\right)\psi(t)\end{aligned}$$

The time dependent Schrödinger equation reads as:

$$\hbar\left(i\frac{1}{2}(\dot{\alpha}^*\alpha - \alpha^*\dot{\alpha}) - \dot{\beta} - \dot{\alpha}^* a^+ - \dot{\alpha} a\right)\psi(t) = -\hbar\omega\left(\frac{d(t)}{\sqrt{2}x_0}(ae^{-i\omega t} + a^+ e^{i\omega t}) - \frac{1}{2}\frac{d^2(t)}{x_0^2}\right)\psi(t)$$

$$\dot{\alpha} = \frac{d(t)}{\sqrt{2}x_0}\omega e^{-i\omega t} \ , \quad \dot{\alpha}^* = \frac{d(t)}{\sqrt{2}x_0}\omega e^{i\omega t} \ , \quad \dot{\beta} = \frac{i}{2}(\dot{\alpha}^*\alpha - \alpha^*\dot{\alpha}) - \frac{\omega}{2}\frac{d^2(t)}{x_0^2}$$

The α and β functions can be easily obtained by integration:

$$\alpha(t) = \frac{\omega}{\sqrt{2}x_0}\int_0^t d(t')e^{-i\omega t'}dt' \ , \quad \beta(t) = \int_0^t \left(\frac{i}{2}(\dot{\alpha}^*\alpha - \alpha^*\dot{\alpha}) - \frac{\omega}{2}\frac{d^2(t')}{x_0^2}\right) dt'$$

Polarization:

In the following example the $d(t)$ shift of the harmonic oscillator depends on the time as:

$$d(t) = \begin{cases} d_0 e^{\gamma t}\cos(\omega_0 t) & \text{if} \quad t < 0 \\ d_0\cos(\omega_0 t) & \text{if} \quad t \geq 0 \end{cases}$$

$$\alpha^*(t) = \frac{d\omega}{\sqrt{2}x_0}\frac{1}{2}\left(\int_{-\infty}^0 \left(e^{(\gamma+i\omega_0)t} + e^{(\gamma-i\omega_0)t}\right)e^{i\omega t}dt + \int_0^t \left(e^{i\omega_0 t'} + e^{-i\omega_0 t'}\right)e^{i\omega t'}dt'\right)$$

Evaluete the first integral:

$$\frac{1}{2}\int_{-\infty}^0 \left(e^{(\gamma+i(\omega+\omega_0)t} + e^{(\gamma+i(\omega-\omega_0)t}\right)dt = \frac{1}{2}\left(\frac{1}{\gamma+i(\omega+\omega_0)} + \frac{1}{\gamma+i(\omega-\omega_0)}\right)$$

In case of adiabatic turning on of the time dependent potential we should take the $\gamma \to 0$ limit:

$$\lim_{\gamma\to 0}\frac{1}{2}\int_{-\infty}^0 \left(e^{(\gamma+i(\omega+\omega_0)t} + e^{(\gamma+i(\omega-\omega_0)t}\right)dt = \frac{i\omega}{\omega_0^2-\omega^2}$$

The second integral:

$$\frac{1}{2}\int_0^t \left(e^{i(\omega_0+\omega)t'} + e^{(\omega-i\omega_0)t'}\right)dt' = -\frac{i\omega}{\omega_0^2-\omega^2} + e^{i\omega t}\frac{-i\omega\cos(\omega_0 t)+\omega_0\sin(\omega_0 t)}{\omega^2-\omega_0^2}$$

$$\alpha^*(t) = \frac{d_0\omega}{\sqrt{2}x_0}e^{i\omega t}\frac{-i\omega\cos(\omega_0 t)+\omega_0\sin(\omega_0 t)}{\omega^2-\omega_0^2}$$

Suppose that at the beginning the harmonic oscillator is at ground state:

$$\psi(t) = e^{i(\alpha(t)^*a^+ + \alpha(t)a + \beta(t))}\varphi_0 = e^{-\frac{|\alpha|^2}{2}+i\beta}e^{i\alpha^*a^+}e^{i\alpha a}\varphi_0 = e^{i\beta}e^{-\frac{|\alpha|^2}{2}}e^{i\alpha^*a^+}\varphi_0 \ .$$

The time dependent solution of the Schrödinger equation in interaction picture is a coherent state[1]:

$$\psi(t) = e^{i\beta-\frac{|\alpha|^2}{2}}e^{i\alpha^*a^+}\varphi_0 = e^{i\beta}|i\alpha^*\rangle \ .$$

First find the expectation value of the position operator:

$$\langle\psi(t)|x|\psi(t)\rangle = \langle i\alpha^*|\frac{x_0}{\sqrt{2}}ae^{-i\omega t} + a^+e^{i\omega t}|i\alpha^*\rangle = \frac{x_0}{\sqrt{2}}\left(e^{-i\omega t}i\alpha^* - ie^{i\omega t}\alpha\right)$$

$$\langle\psi(t)|x|\psi(t)\rangle = \frac{d_0\omega}{2}\left(\frac{\omega\cos(\omega_0 t)+i\omega_0\sin(\omega_0 t)}{\omega^2-\omega_0^2} - \frac{-\omega\cos(\omega_0 t)+i\omega_0\sin(\omega_0 t)}{\omega^2-\omega_0^2}\right)$$
$$= d_0\frac{\omega^2}{\omega^2-\omega_0^2}\cos(\omega_0 t)$$

[1]**Coherent state of a harmonic oscillator:**

The coherent state of a harmonic oscillator is an eigen state of the lowering operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

where $\alpha \in \mathcal{C}$, since a is a non-hermitian operator. Consider the following wavefunction:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^+}|\varphi_0\rangle \ .$$

$$a|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} a e^{\alpha a^+}|\varphi_0\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^+} a|\varphi_0\rangle + e^{-\frac{1}{2}|\alpha|^2}[a, e^{\alpha a^+}]|\varphi_0\rangle \ .$$

Obviously $a|\varphi_0\rangle = 0$ and the commutator has been determined already in the previous section: $[a, e^{\alpha a^+}] = \alpha e^{\alpha a^+}$. We got an eigen value equation:

$$a|\alpha\rangle = a e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^+}|\varphi_0\rangle = \alpha e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^+}|\varphi_0\rangle$$

We should also check the norm of the state: $\langle\alpha|\alpha\rangle = e^{-|\alpha|^2}\langle\varphi_0|e^{\alpha^* a} e^{\alpha a^+}|\varphi_0\rangle$. Use the Baker - Campbell - Hausdorff expansion:

$$e^{\alpha^* a} e^{\alpha a^+} = e^{\alpha^* a + \alpha a^+ + \frac{1}{2}|\alpha|^2 [a, a^+]} = e^{\alpha^* a + \alpha a^+ + \frac{1}{2}|\alpha|^2} \ , \qquad e^{\alpha a^+} e^{\alpha^* a} = e^{\alpha^* a + \alpha a^+ + \frac{1}{2}|\alpha|^2 [a^+, a]} = e^{\alpha^* a + \alpha a^+ - \frac{1}{2}|\alpha|^2}$$

it is straightforward to see, that

$$e^{\alpha^* a} e^{\alpha a^+} = e^{|\alpha|^2} e^{\alpha a^+} e^{\alpha^* a} \ .$$

Use this relation in the expression of the norm and keep in mind that $\langle\varphi_0|e^{\alpha a^+} = \langle\varphi_0|$ and $e^{\alpha^* a}|\varphi_0\rangle = |\varphi_0\rangle$:

$$\langle\alpha|\alpha\rangle = e^{-|\alpha|^2}\langle\varphi_0|e^{\alpha^* a} e^{\alpha a^+}|\varphi_0\rangle = \langle\varphi_0|\varphi_0\rangle = 1$$