Commutator of the Hamiltonian of He atom and $L_1 + L_2$ angular momentum operator

The Hamilton operator of the He atom can be written as:

$$H = \underbrace{\frac{\mathbf{p}_1^2}{2m} - \frac{2ke^2}{r_1}}_{H(1)} + \underbrace{\frac{\mathbf{p}_2^2}{2m} - \frac{2ke^2}{r_2}}_{H(2)} + \underbrace{\frac{ke^2}{|\mathbf{r}_1 - \mathbf{r}_2|}}_{H_{12}} .$$

L_1 and L_2 obviously commute with $H(1)$ and $H(2)$ they contain a spherically symmetric potential. We have to prove that the Coulomb repulsion between the electrons also commutes with $L_1 + L_2$. We will prove that L_z commutes with the repulsion potential. Due to spherical symmetry the chosen of the orientation of the z axis is arbitrary, consequently, if $[L_z, H] = 0$ all the components of the angular momentum operator will commute with the Hamiltonian. The L_z operator in spherical coordinates: $L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$.

$$\begin{aligned}
&[L_{1z} + L_{2z}, \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}]\Psi = (L_{1z} + L_{2z})\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}\Psi - \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}(L_{1z} + L_{2z})\Psi \\
&= \frac{\hbar}{i}\left(\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\right)\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}\Psi - \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}\left(\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\right)\Psi \\
&= \Psi\frac{\hbar}{i}\left(\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\right)\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} + \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}\left(\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\right)\Psi - \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}\left(\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\right)\Psi \\
&= \Psi\frac{\hbar}{i}\left(\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\right)\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}
\end{aligned} \tag{1}$$

In order to evaluate the derivatives we will expand $\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}$ using Legende polynomials:

$$\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\hat{\mathbf{r}_1}\hat{\mathbf{r}_2}) ,$$

where $r_<$ denotes the length of the shorter of $|\mathbf{r}_1|$ and $|\mathbf{r}_2|$ and $r_>$ denotes the length of the longer of $|\mathbf{r}_1|$ and $|\mathbf{r}_2|$, $\hat{\mathbf{r}}_1$, $\hat{\mathbf{r}}_2$ are unit vectors $\hat{\mathbf{r}}_1 = \frac{\mathbf{r}_1}{r_1}$ and $\hat{\mathbf{r}}_2 = \frac{\mathbf{r}_2}{r_2}$. The scalar product of the unit vectors can be given as:

$$\hat{\mathbf{r}}_1\hat{\mathbf{r}}_2 = \sin(\vartheta_1)\sin(\vartheta_2)\cos(\varphi_1 - \varphi_2) + \cos(\vartheta_1)\cos(\vartheta_2)$$

Calculate the derivatives with respect of φ_1 and φ_2:

$$\begin{aligned}
\frac{\partial}{\partial\varphi_1}\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} &= -\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l`(\hat{\mathbf{r}_1}\hat{\mathbf{r}_2})\sin(\vartheta_1)\sin(\vartheta_2)sin(\varphi_1 - \varphi_2) \\
\frac{\partial}{\partial\varphi_2}\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} &= \sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l`(\hat{\mathbf{r}_1}\hat{\mathbf{r}_2})\sin(\vartheta_1)\sin(\vartheta_2)sin(\varphi_1 - \varphi_2) = -\frac{\partial}{\partial\varphi_1}\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|}
\end{aligned}$$

Easy to se that the last term in Eq.1 will be zero:

$$\frac{\hbar}{i}\left(\frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\right)\frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = 0$$

We saw that the commutator of the $L_{1z} + L_{2z}$ operator with the Hamiltonian of the He atom is zero. Due to the spherical symmetry all components of $L1\alpha) + L_{2\alpha}$ wher $\alpha = x, y, z$ will commute with the Hamiltonian and $(L_{1z} + L_{2z})^2$ total angular momentum square also commutes with it:

$$[J_i, H_{He}] = 0 , \quad i = 1, 2, 3 , \qquad [J^2, H_{He}] = 0$$