1. gyakorlat (szept. 6.)

Clebsh-Gordan cofficients

1 $l_1 = 1,\ l_2 = 1$

Consider an excited state of a Helium atom where both electrons are on a $2p$ orbital. Construct the states belonging to different total angular momenta!

The total angular momenta of the system are: $|l_1 - l_2| \geq J \geq l1 + l2$, in our particular case J takes the following values $J = 0, 1, 2$. The state belonging to the largest angular momentum can be easily constructed as a product state:

$$|2,2\rangle = |1,1\rangle|1,1\rangle$$

The other $J = 2$ states can be constructed by applying the lowering operator on both sides of the previous equation:

$$\begin{aligned} J_-|2,2\rangle &= (l_{1-} + l_{2-})|1,1\rangle|1,1\rangle \\ 2|2,1\rangle &= \sqrt{2}(|1,0\rangle|1,1\rangle + |1,1\rangle|1,0\rangle) \\ |2,1\rangle &= \frac{1}{\sqrt{2}}\left(|1,0\rangle|1,1\rangle + |1,1\rangle|1,0\rangle\right) \end{aligned}$$

The $|1,1\rangle$ state must be orthogonal to the $|2,1\rangle$ obtained in the previous equation. In order to get an orthogonal state we can simply change the sign in the product state $|2,1\rangle$:

$$|1,1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle|1,1\rangle - |1,1\rangle|1,0\rangle\right)$$

Let us apply further the lowering operator:

$$\begin{aligned} J_-|2,1\rangle &= (l_{1-} + l_{2-})\frac{1}{\sqrt{2}}\left(|1,0\rangle|1,1\rangle + |1,1\rangle|1,0\rangle\right) \\ \sqrt{6}|2,0\rangle &= \frac{1}{\sqrt{2}}\left(\sqrt{2}|1,-1\rangle|1,1\rangle + \sqrt{2}|1,0\rangle|1,0\rangle + \sqrt{2}|10\rangle|10\rangle + \sqrt{2}|1,1\rangle|1,-1\rangle\right) \\ J_-|1,1\rangle &= (l_{1-} + l_{2-})\frac{1}{\sqrt{2}}\left(|1,0\rangle|1,1\rangle - |1,1\rangle|1,0\rangle\right) \\ \sqrt{2}|1,0\rangle &= \frac{1}{\sqrt{2}}\left(\sqrt{2}|1,-1\rangle|1,1\rangle + \sqrt{2}|1,0\rangle|1,0\rangle - \sqrt{2}|10\rangle|10\rangle - \sqrt{2}|1,1\rangle|1,-1\rangle\right) \end{aligned}$$

Summarizing the results:

$$\begin{aligned} |2,0\rangle &= \frac{1}{\sqrt{6}}\left(|1,-1\rangle|1,1\rangle + 2|1,0\rangle|1,0\rangle + |1,1\rangle|1,-1\rangle\right) \\ |1,0\rangle &= \frac{1}{\sqrt{2}}\left(|1,-1\rangle|1,1\rangle - |1,1\rangle|1,-1\rangle\right) \end{aligned}$$

The $|0,0\rangle$ state must be orthogonal both $|2,0\rangle$ and $|1,0\rangle$ states. One can easily prove that

$$|0,0\rangle = \frac{1}{\sqrt{3}}\left(|1,-1\rangle|1,1\rangle - |1,0\rangle|1,0\rangle + |1,1\rangle|1,-1\rangle\right)$$

Homework

In a singlet ($S = 0$) state which total angular momentum can be realized?