Introduction

Random granular media is of practical and theoretical interest. In the literature different level of description is used: either discrete at particle level, where randomness appears among the properties of individual particles, or a continuum approach where coarse-grained macroscopic description is chosen [1-6]. In the following we will adapt the second strategy. However, generally for such continuum approaches, the medium is considered as homogeneous. Continuum models with inhomogeneous properties are uncommon in the literature [7]. Indeed, at a large scale, fluctuations in most macroscopic observations, are generally expected to vanish, and hence considering the medium as homogeneous has no severe consequences.

We consider here a problem of inhomogeneity in granular media from the point of view of continuum mechanics. For non-cohesive granular media submitted to small external loadings, a simple continuum description consists in choosing a rigid-plastic behaviour. The yield surface results from the contact law between individual grains. In the case of simple dry friction, and no texture (isotropic) the yield criterion can be reduced to a simple parameter, the angle of friction. The latter is however not intrinsic and in particular it depends, albeit weakly, on the local compacity. Due to the geometrical constraints (hard-core repulsion) in the packing, compacity is expected to fluctuate from places to places. Therefore at a suitable coarse-grain scale, the medium can be described as a continuum, where the Coulomb angle is a random function displaying small fluctuations around a mean value. The question we address is to compute an effective microscopic Coulomb angle (means and standard deviation) from local distribution.

Our motion is the following: when the sample reaches its macroscopic yield limit, for a homogeneous loading, plastic strain localizes on a shear band. For a homogeneous media this point is a bifurcation point with a very high level of degeneracy. Disorder, of any amplitude, will break this degeneracy and lead to a unique geometry of this first shear band. This extreme sensitivity to disorder suggests that interesting results could be obtained from a perturbative study of this problem including disorder.

Moreover, the fact that the limiting structure is concentrated in space on a narrow domain (slip curve) implies that size effects could be observed. This again can be tested experimentally, at the expense of having reproducible conditions of properties of the sample.

From the theoretical point of view, the above discussed sensibility to disorder (resulting from the non-smoothness of the rigid-plastic behaviour) makes this problem much harder than other problems (smooth mechanics) such as those involving inhomogeneities in elastic models, for which powerful homogenization techniques have been developed (plastic media, self-consistent approaches, bound integrating statistical informations on the structure of the medium, ...).

In Section 1. we present our model of two dimensional random granular media. We submit it to external loadings and we search for the maximum external load that the sample can bear without plastic flow. We propose a first discretization. In section 2. we reformulate the problem into an equivalent one for which we propose two algorithms to solve this simplex maximization problem. This algorithm is similar to the one introduced in the case of a discrete model by S. Ouaguenouni and J. N. Roux [3,4], while the second is based on a different strategy. The result are presented in Section 3. and we report statistical effects similar to those observed critical phenomena.

In the next Section we propose a renormalized group approach that gives a quite different answer with a surprising mapping onto a directed polymer problem from which non-trivial exponents appear. We compare this approach in a limit case to the directed polymer problem on hierarchical lattices and we show the symmetries in the two systems.

The difference obtained in the first approach calls for an explanation. We first show that the type of the lattice used in the first discretization is not responsible for the difference, and explicitly derive a different finite scheme on a triangular lattice which give similar effects.

Another explanation lie in the number of degrees of freedom in the system, which (once balance equations are taken into account) is much less than the number of the constraints due to the local yield limits ($\mathcal{O}(N^2)$).

We thus introduce a different approach based on a potential stress formulation (Airy function). A numerical implementation of the latter problem is described and the first results are presented. At the present stage however the slow convergence of the latter prevents us to derive conclusive statements on the value of the γ exponent and universality class of the problem.

1 Presentation of the first model

Let us consider a rigid two dimensional random granular material. The medium is supposed to be rigid-plastic with local yield criterion of Mohr-Coulomb types with no cohesion and an angle of friction, which is a random function (with a small fluctuating part compared to its mean). Thus locally no plastic flow occurs if:

$$\mu(\mathbf{r}) \leq \frac{\sigma_1(\mathbf{r})}{\sigma_2(\mathbf{r})} \leq \frac{1}{\mu(\mathbf{r})}, \tag{1}$$

where $\sigma_1(\mathbf{r})$, $\sigma_2(\mathbf{r})$ denote the two eigenvalues of the stress tensor and $\mu(\mathbf{r})$ the static friction coefficient at point $\mathbf{r}$.

We are interested in the maximal possible external weight that this material can balance without plastic flow. Theoretically necessary condition for the material to support the load is that one can find at least one acceptable solution for the stress tensor $\underline{\underline{\sigma}}(\mathbf{r})$ that satisfies in every point the Coulomb-criterion.

The stress tensor must satisfy the stress continuity equations. Here we suppose that the weight of the granular material is negligible compared to the external load. The stress continuity equations are the following

$$\partial_x \sigma^{xx} + \partial_y \sigma^{xy} = 0 \tag{2a}$$
$$\partial_x \sigma^{xy} + \partial_y \sigma^{yy} = 0, \tag{2b}$$

where ∂_x (∂_y) denotes the partial derivation in the x (y) direction, the tensor $\underline{\underline{\sigma}}$ is symmetric so $\sigma^{xy} = \sigma^{yx}$. Boundary conditions are such that the external stress field i.e. Σ_1 horizontally and Σ_2 vertically is balanced by the resulting mean stress on each boundary. A solution is admissible if it fulfills the boundary conditions the balance (1) and the continuity equations (2).

At a fixed Σ_1 we search for the maximal value of Σ_2 for which there exists at least one admissible solution.

In order to solve this problem we discretize the model. We split a square domain in $N \times N$ equal cells (Fig. 1). We suppose that in every lattice site the stress tensor and the static friction coefficient is constant. The balance equations (2) in this discretized version are satisfied inside the sites and between the cells at the common boundaries the x and the y projections of

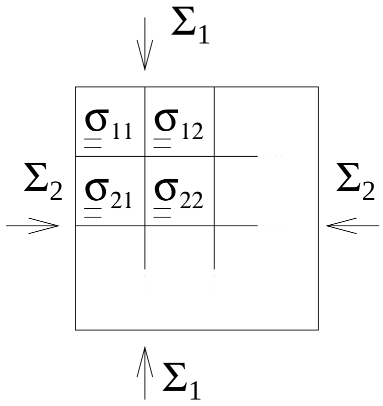

Figure 1: Discretization of the sample and displacement of the forces.

the stress vectors must be equal. On the outer boundaries

$$a \sum_{i=1}^{N} \underline{\underline{\sigma}}_i \, \underline{n} = L \underline{\underline{\Sigma}} \, \underline{n},$$

where n is the normal vector of the boundary, a is the linear length of a discretization site and $L = Na$ is the linear length of the whole sample so $\langle \underline{\underline{\sigma}}_i \, \underline{n} \rangle_i = \underline{\underline{\Sigma}} \, \underline{n}$.

Let us denote by Σ_2 (Σ_1) the external normal forces that are applied on the vertical (horizontal) boundaries. The external boundary conditions impose:

$$\text{Horizontally :} \quad \frac{1}{N} \sum_{k=1}^{N} \sigma_{ik}^{yy} = \Sigma_1 \quad \text{and} \tag{3a}$$

$$\sum_{k=1}^{N} \sigma_{ik}^{xy} = 0 \tag{3b}$$

$$\text{Vertically :} \quad \frac{1}{N} \sum_{i=1}^{N} \sigma_{ik}^{xx} = \Sigma_2 \quad \text{and} \tag{3c}$$

$$\sum_{i=1}^{N} \sigma_{ik}^{xy} = 0 \tag{3d}$$

$$(i, k = 1, N)$$

Boundary conditions between cells assume the following form:

$$
\begin{aligned}
\text{Horizontally}: \quad \sigma_{ki}^{yy} &= \sigma_{k+1,i}^{yy} \quad \text{and} && (4a)\\
\sigma_{ki}^{xy} &= \sigma_{k+1,i}^{xy} && (4b)\\
\text{Vertically}: \quad \sigma_{ik}^{xx} &= \sigma_{i,k+1}^{xx} \quad \text{and} && (4c)\\
\sigma_{ik}^{xy} &= \sigma_{i,k+1}^{xy} && (4d)
\end{aligned}
$$

$$\text{for all } k = 1, \ldots, N-1 \quad \text{and} \quad i = 1, \ldots, N.$$

From the inner boundary conditions (4b) and (4d) we can conclude that $\sigma_{ij}^{xy} = Const.$ for any (ij) pair. This constant is zero because of the external boundary conditions (3b) and (3d) (there is no shear). Therefore in all cells the principle stress directions are parallel to the x and y axis. This reduces the Coulomb-criterion to the simple $\sigma_{ij}^{xx}/\sigma_{ij}^{yy}$ form. The inner boundary conditions imply that in a row (column) the xx (yy) component of the stress tensor is constant:

$$
\begin{aligned}
\sigma_{ij}^{xx} &= \sigma_i^{xx} \quad (\text{for all } j = 1, 2, \ldots, N)\\
\sigma_{ij}^{yy} &= \sigma_j^{yy} \quad (\text{for all } i = 1, 2, \ldots, N)
\end{aligned}
$$

The external boundary conditions (3a) and (3c) give the sum of these variables:

$$\frac{1}{N}\sum_{i=1}^{N} \sigma_i^{xx} = \Sigma_2, \quad \frac{1}{N}\sum_{i=1}^{N} \sigma_i^{yy} = \Sigma_1. \tag{5}$$

For an imposed Σ_1, Σ_2 there are only $2N-2$ unknowns

$$\sigma_i^{xx}, \quad \text{for } i = 1, 2, \ldots N-1 \quad \text{and} \quad \sigma_j^{yy}, \quad \text{for } j = 1, 2, \ldots N-1.$$

Searching for the maximum of Σ_2 with fixed Σ_1 means that there are $2N-1$ free variables and thus $2N-1$ inequalities can be fulfilled.

One can note that this is a "simplex" linear optimization problem. This means that we need to maximize a variable Σ_2 which is one of the free variables. Every variable is positive (primary constraints), and there are additional constraints that some linear combinations of these variables are larger or smaller than a non-negative value (b_i)

$$a_{i,1}\sigma_1^{xx} + \cdots + a_{i,N}\sigma_N^{xx} + a_{i,N+1}\sigma_1^{yy} + \cdots + a_{1,2N-1}\sigma_{N-1}^{yy} \leq b_i \ (\geq b_i)$$

The Coulomb equalities (1) obviously match this form,

$$\mu_{ij}\sigma_i^{xx} - \sigma_j^{yy} \leq 0$$

so this is really a simplex problem.

However, we will not use the standard simplex algorithm [8] to solve it. The reason for it is the large $2N^2$ number of inequalities, which implies that the standard implementation of the simplex algorithm is very time-consuming. The *tableau* of the algorithm has the dimension of at the order $2N^2 \times 2N^2$ that gives high, at least N^4 size dependence. The algorithm we introduced solve the problem faster ($\mathcal{O}(N^{3.5})$). With this algorithm we could solve the problem for system sizes up to 200×200.

We introduce some conventions that we will use in the rest of this work. One notes that we can multiply every stress variable by an arbitrary constant without any change. So we set Σ_1 to 1 and we maximize the variable Σ_2. We denote σ^{xx} by σ^x and the same for the yy component since the shear component is null.

2 Linear algorithm for the model

2.1 Notations and representation

Following the idea of S. Ouaguenouni and J. N. Roux we transform the optimization problem into an equivalent one for accessing the solution numerically more efficiently than with the simplex method.

The representation where we have the same σ^y in each row and the same σ^x in each column ensures that the balance equations are automatically fulfilled. We should deal only with the yield criteria.

It is more convenient to work with other variables than σ_i^x and σ_i^y. We will use their partial sums

$$s_i^x \equiv \sum_{j=1}^{i} \sigma_j^x, \quad \text{and} \quad s_i^y \equiv \sum_{j=1}^{i} \sigma_j^y.$$

Trivially inverted

$$\sigma_i^x = s_i^x - s_{i-1}^x \quad \text{and} \quad \sigma_i^y = s_j^y - s_{j-1}^y$$

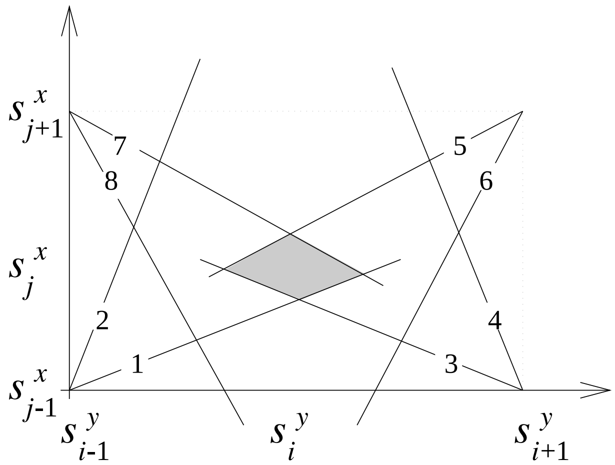

Figure 2: In the current situation the point (s^x_i, s^y_j) can be anywhere in the filled space. The slope of the lines are respectively μ_{ij}, $1/\mu_{ij}$, $\mu_{i+1,j}$, $1/\mu_{i+1,j}$, $\mu_{i+1,j+1}$, $1/\mu_{i+1,j+1}$, $\mu_{j+1,i}$, $1/\mu_{j+1,i}$

Conventionally we set: $s^x_0 \equiv s^y_0 \equiv 0$.

One important consequence of these definitions is that if we change the value of s^x_i we modify both σ^x_i and σ^x_{i+1}. The same applies for the y component.

The convenience of these definitions is that $s^x_i \leq s^x_{i+1}$ and $s^y_i \leq s^y_{i+1}$ for all i because all σ^x_i and σ^y_i are larger than zero. If we represent them in a coordinate system they will be well ordered.

We express the Coulomb-criteria with the variables s^x_i and s^y_j:

$$\mu_{ij} \leq \frac{s^y_j - s^y_{j-1}}{s^x_i - s^x_{i-1}} \leq \frac{1}{\mu_{ij}}.$$

In the coordinate systems it means that the point (s^x_i, s^y_j) must lie between the two lines passing through the point (s^x_{i-1}, s^y_{j-1}) with slope μ_{ij} and $1/\mu_{ij}$ (Fig. 2). As we mentioned the position of the (s^x_i, s^y_j) point influences the value of four σ: σ^x_i, σ^x_{i+1}, σ^y_j and σ^y_{j+1}. So the same construction applies at these points too. Thus the (s^x_i, s^y_j) point must lie in a quadrilateral, shaded in gray in Fig. 2.

Initially we take the configuration

$$\sigma^x_i = \sigma^y_i = 1 \quad \Longleftrightarrow \quad s^x_i = s^y_i = i$$

that satisfies all Coulomb-criteria for any values of $0 < \mu_{ij} < 1$ so every point lies in the permitted space.

If we increase Σ_2 the size of these domains gets smaller to the limit where some of them become a point. If we still increase Σ_2 we arrive at a state where we cannot find any acceptable solutions. Our algorithm is based on this observation. We start from an acceptable configuration, we increase Σ_2 until we reach the point where we cannot increase Σ_2 without violating the Coulomb-criterion.

In order to keep track of an acceptable solution for the stresses as Σ_2 increases, we introduce forces which repel the configuration point (s) into the admissible domain as it approaches the Coulomb-criterion. Moreover as we wish to reach an equilibrium state we introduce a viscous dissipation. So we took viscous forces that are proportional to the velocity and with a magnitude that diverges as the point gets closer to a line. In our model these forces were perpendicular to the Coulomb-criteria lines in order to allow a free motion along the boundaries. Moreover we exert an external "force" to increase Σ_2. The stress field represented as a point in this space is followed by solving for the mechanical evolution of this system.

Unfortunately, usual relaxation dynamics is too slow to converge to the good solution. Usually the points are trapped in corners from which they cannot escape. This happens usually because the lines coming from the opposite corners have the slope μ_{ij} and $\mu_{i+1,j+1}$ that can be very close to each other for small disorder. The intersection of these lines moves much faster than the lines itself and it easily catches up the point between them (Fig. 3). In other words the system becomes ill conditioned and requires either an implicit coding a small time stepping which become prohibitively slow.

2.2 Another strategy

In order to speed up the convergence and to avoid this corner effect we introduce a different strategy. The dissipation potential is arbitrary. We can even take a hard core potential ($0 - \infty$ valued), reverting back to a non-smooth law. We replace the $F(v)$ force-velocity function by this law (see Fig. 4).

At the beginning every point is far from the Coulomb-criteria lines. In

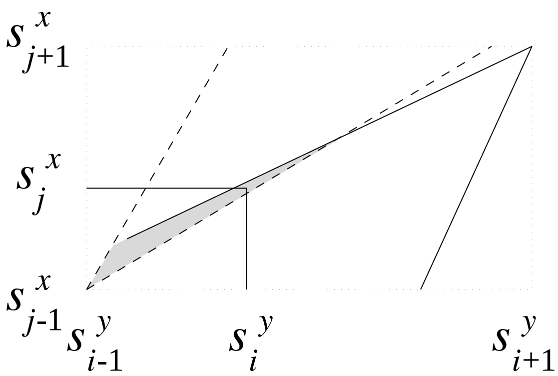

Figure 3: A point can be easily trapped by two lines with slightly different slope. The shaded area is admissible for the point (s^y_i, s^x_j).

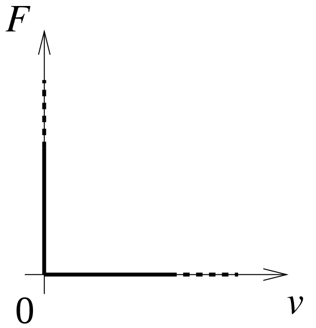

Figure 4: The relation between the velocity and the force. If the force is zero the velocity can be anything. If the velocity is zero the force can have any value.

this case the lack of external forces means that the sum of the forces must be zero in each point. In this case the velocity is unknown so we calculate. We get velocities at each point and now we advance our system. The time step is arbitrary so we take the shortest time (as in the Event Driven mechanics) needed for a point to reach a line i.e. to satisfy a Coulomb-criterion. For a point that is on a line there must be a constraint that keeps fixes it to the line. This gives its velocity relative to the others but now the force is unknown so we have to calculated it. If the force points outward from the permitted space it keeps the point on the Coulomb-criterion line but in the opposite case we should release the point calculating again its velocity.

Repeating the above procedure we arrive at a state where we have $2N-1$ points on the lines namely we have $2N-1$ satisfied Coulomb-criteria. At this point we check once more that every point-line contact (satisfied Coulomb-criterion) is stable and stop if this is true. At this point we have found the maximal possible value of Σ_2. We have no more free variables and the existing contacts are stable so Σ_2 cannot increase any longer. We have found the solution.

3 Results

3.1 Numerical results

The simulations were done as described in the previous chapter. The values of μ_{ij} are distributed randomly with a uniform distribution in the interval $[a, b]$, where $0 < a < b < 1$. With the algorithm, described in the previous section, we find the maximum of Σ_2 for given initial μ_{ij} configurations. We did the simulations on middle size workstations. We could examine system sizes up to 200. We record the value of the Σ_2 and the positions of the satisfied Coulomb-criterion and we average over many samples (20-10000).

The value of $\Sigma_2(N)$ decreases as the system size increases. For large values of N this behaviour can be characterized by an asymptotic constant and a power law

$$\Sigma_2(N) = \Sigma_\infty + cN^{-\gamma}. \tag{6}$$

This formula is reminiscent from critical phenomena. However we will see that in the present case the critical point Σ_∞ and the exponent γ are trivial.

We will see in the further section another approach leading to a formally similar behaviour as Eq. (6) with however a non trivial critical exponent γ, which is the signature of a genuine critical point which will be identified to one classical problem of statistical physics i.e. conformation of directed polymers in a random potential at zero temperature.

Normally the fit of the constant Σ_∞ is a crucial point because the exponent γ depends very strongly on it. Fortunately, it turned out that in our case this constant is very simple: $\Sigma_\infty = 1/b$. The exponent γ is found to be $\gamma = 1 \pm 0.03$.

The multiplicative factor c depends only on the initial parameters, namely on a and on b. We observed numerically (and we will justify in the following) that c can be written in the following from

$$c = \frac{b-a}{b^2}\ \alpha,$$

where α is a number. Hence we have

$$\langle \Sigma_2(N) \rangle = \frac{1}{b} + \frac{(b-a)}{b^2}\ \frac{\alpha}{N} \tag{7}$$

Thus if we plot the following function we can get rid of the dependence on the initial parameters:

$$\omega = \left(\langle \Sigma_2 - \frac{1}{b} \right) \frac{b^2}{b-a} \tag{8}$$

The Fig. 5 shows ω versus N in log-log scale the expected $\omega = \alpha/N$ relation is plotted as a solid line and the data plots obtained for different $[a, b]$ values. In our case the numerical value for α is $\alpha \simeq 1.6$.

3.2 Approximation by a simple model

These properties of the system can be understood very easily. At the final stage of our model we must satisfy at least one Coulomb-criterion in each line and column. If it is not true we can increase and decrease the value of the corresponding σ_i^x or σ_i^y without violating any Coulomb-criteria. Accordingly we increase the ratio of Σ_2/Σ_1 what is contrary to the initial hypothesis of an optimal solution for Σ_2/Σ_1.So there must be a satisfied Coulomb-criterion in each row and column.

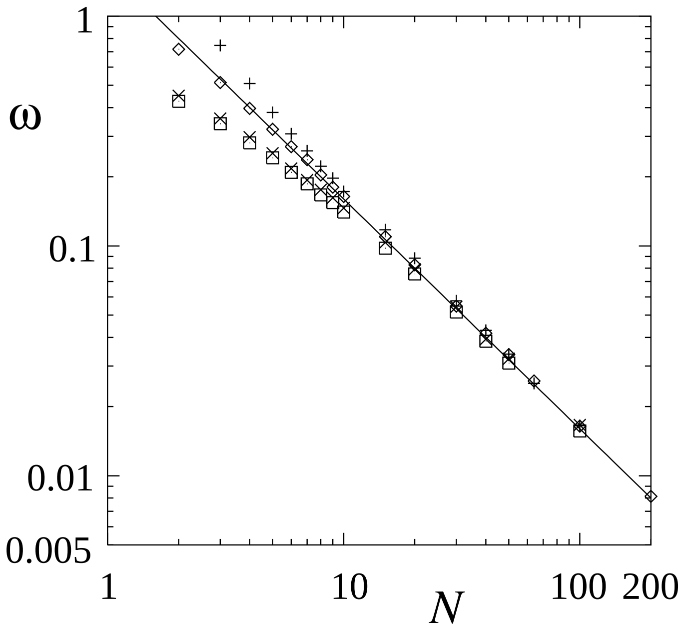

Figure 5: The size dependence of the normalized (8) maximal possible external deviatoric stress on log-log plot. The symbols represent different intervals for μ_{ij}: $+$: [0,1], $\diamond$: [0.2,0.8], $\Box$: [0.8,1], $\times$: [0.5,0.7]. The solid line is the $1.6/N$ function.

We have $2N-1$ satisfied Coulomb-criteria. It means that on average two satisfied criteria can be found in each line and column. Our approach is the following. We start from the same initial configuration as for the numerical simulations i.e. $\sigma_i^x = \sigma_i^y = 1$. We take each row and we increase each σ_i^x to satisfy a Coulomb-criterion in this row, but do not violate any. In this way we increase Σ_2. After this step usually there are columns where there is no satisfied Coulomb-criterion. Here we decrease the σ_i^y in these columns to satisfy a Coulomb-criterion and to decrease Σ_1. We get an approximation of the maximal Σ_2/Σ_1 in this way.

In every row we satisfy one Coulomb-criterion. This is where μ_{ij} is the largest because it can bear the least weight. The expectation value of the largest μ_{ij} in a row is

$$\mu_l = b - (b-a)/N.$$

This let us increase $\Sigma_2 = \sum_{i=0}^{N} \sigma_i^x/N$ equ. (5) from 1 to the value:

$$\Sigma_2 = \frac{1}{\mu_l} = \frac{1}{b}\left(1 + \left(\frac{b-a}{b}\right)\frac{1}{N}\right) \tag{9}$$

i.e. as in Eq. (7). We get a lower limit:

$$\alpha = 1. \tag{10}$$

An upper limit can be obtained considering that there are $2N-1$ satisfied Coulomb-criteria. It is an upper limit for α since the set of satisfied yield criteria may not be completed with this. When we satisfy one criterion in each column only the second largest μ_{ij} is available because the largest was satisfied when the rows were considered.

$$\mu_{l-1} = b - 2(b-a)/N.$$

This assumption gives the upper limit:

$$\alpha = 2. \tag{11}$$

We can give a better lower approximation considering the fact that after we satisfy one Coulomb-criteria in each row there are some free columns and by modifying σ_x only in these columns we will not violate any Coulomb-criterion. This will be a lower limit because there remains some free variables and by changing them we can still increase Σ_2/Σ_1 without violating any Coulomb-criteria.

The expectation value of the number of the free columns is detailed in Appendix A and amounts to

$$E_N(N) = \left(1 - \frac{1}{N}\right)^{N-1} (N-1) \quad \xrightarrow[N\to\infty]{} \quad \frac{N}{e} \tag{12}$$

We use the previous value of μ_{l-1} and we get the lower limit:

$$\alpha = \frac{N + (N-1)\left(1-\frac{1}{N}\right)^{N-1}}{(N-1) + \frac{a}{b} - \frac{N-1}{N}\left(1-\frac{a}{b}\right)\left(1-\frac{1}{N}\right)^{N-1}} \quad \xrightarrow[N\to\infty]{} \quad 1 + \frac{1}{e} \tag{13}$$

This lower limit ($\alpha \simeq 1.37$) is just 15% bellow the measured value ($\alpha \simeq 1.6$) and it gives also some small small size effect: Fig. 6.

The good correspondence of these simple models with the measured values is due to the very small ($\mathcal{O}(N)$) number of degrees of freedom compared to the total constraints. That is the reason of the trivial Σ_∞ and the $\gamma = 1$ exponent resulting from extreme value statistics. In order to find something more interesting, we should have more ($\mathcal{O}(N^2)$) degrees of freedom. Thus $\gamma = 1$ is *not* universal but merely reflects the slope of the distribution of μ close to its maximal value.

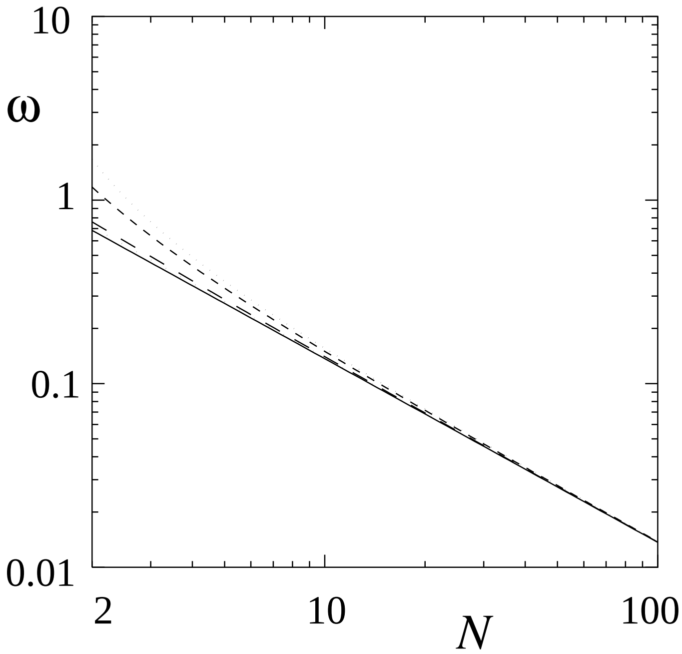

Figure 6: The size dependence of the approximated lower limit of the maximal possible deviatoric stress using Eq. (13). The lowest solid line is the asymptotic curve: $(1+1/e)/N$, the other curves are generated using Eq. (13) with the same parameters as on Fig. 5. From down to up $[a,b]$ is the following: [0.8,1], [0.5,0.7], [0.2,0.8] and [0,1].

4 Renormalization group approach

Let us consider a small 2×2 system, where the problem can be solved exactly. We have three free variables and we can fulfill three Coulomb-criteria out of four. We chose three free variables: Σ_2, σ_1^x and σ_2^y. The other stress components can be expressed in the following way:

$$\sigma_2^x = \Sigma_2 - \sigma_1^x \quad \text{and} \quad \sigma_2^y = 1 - \sigma_1^y.$$

For simplicity let us denote by a, b, c and d the four μ_{ij} respectively. Σ_2 can be easily calculated for the four possible configurations.

$$\text{if } bc > ad \qquad \Sigma_2 = \max\left(\frac{a+b}{b(a+c)}, \frac{c+d}{c(b+d)}\right) \tag{14a}$$

$$\text{if } bc < ad \qquad \Sigma_2 = \max\left(\frac{a+b}{a(b+d)}, \frac{c+d}{d(a+c)}\right) \tag{14b}$$

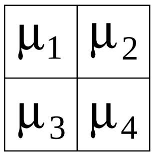

Figure 7: The representation of the friction coefficients in the 2×2 system.

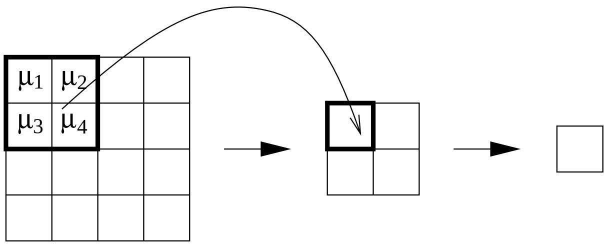

Figure 8: We solve the problem for 2×2 systems. In this system we get the optimal value for Σ_2. We take this value as a new $1/\mu$ and we redo the procedure but with a half as large system.

We can ease our calculations by the fact that the result of the max function is determined in a very simple way. In the equation (14a) if $b < c$, the second term is always larger and vice versa. The same applies for (14b). Therefore the configuration that maximizes Σ_2 depends only on the friction coefficients in the diagonal, where their product is larger.

We perform a real space renormalization analysis in the following way. We take a large $N = 2^k$ size system. For every 2×2 square we solve our problem exactly. Replacing the achieved Σ_2 into $1/\mu$ we reduce the linear system size by a factor of two (see Fig. 8. We repeat the above procedure until we get only one Σ_2 when we get the final result.

We suppose that the size dependence of the renormalized Σ_{2r} follows the same law as before (6)

$$\langle \Sigma_2(N) \rangle = \Sigma_\infty + bN^{-\gamma}.$$

As one can observe on Fig. 9 the law is verified over more than two decades. The normalization of the curves was done with the same function as in Eq. (8).

The scaling is valid but there are main differences respect to the previous model. There the number of the satisfied Coulomb-criteria was proportional to N while here this is $3N^2/4 \sim N^2$. This gives a nontrivial value for

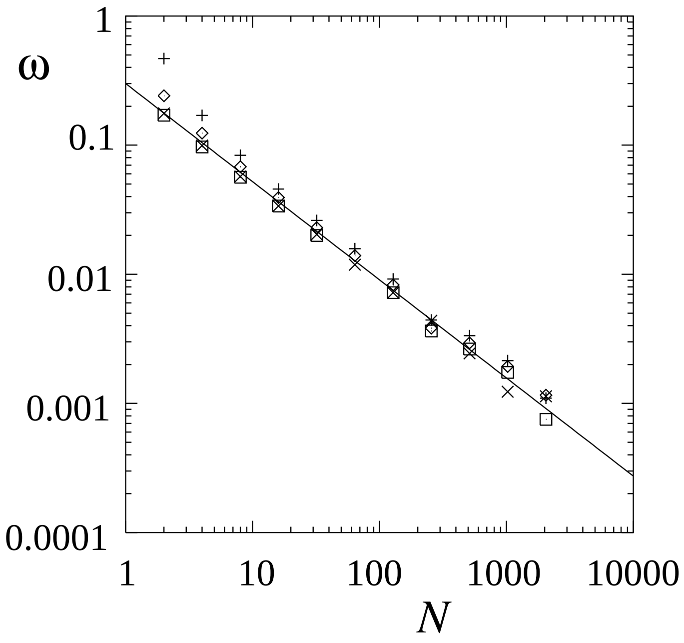

Figure 9: The size dependence of the normalized (8) maximal possible external deviatoric stress calculated by renormalization. The symbols represent different intervals for μ_{ij}: $+$: [0,1], $\diamond$: [0.2,0.8], $\square$: [0.8,1], $\times$: [0.5,0.7]. The solid line is the $0.3N^{-0.76}$ function.

$1/a < \Sigma_\infty < 1/b$ and we get also a non-trivial exponent $\gamma = -0.76 \pm 0.05$.

One can try to calculate exactly the renormalized Σ_{2r}. Unfortunately, only the inverse of the $\Sigma_{2r}(N)$ can be calculated exactly and only for $N = 2$. This reads in the following formula

$$\left\langle \frac{1}{\Sigma_{2r}} \right\rangle = \frac{\displaystyle\int_0^1 \mathrm{d}d \int_0^1 \mathrm{d}a \int_{\sqrt{ad}}^1 \mathrm{d}c \int_{ad/c}^c \frac{a+b}{a+c} c \;\mathrm{d}b}{\displaystyle\int_0^1 \mathrm{d}d \int_0^1 \mathrm{d}a \int_{\sqrt{ad}}^1 \mathrm{d}c \int_{ad/c}^c 1 \;\mathrm{d}b} = \frac{529}{360} - \frac{4\ln 2}{3} \simeq 0.545.$$

This values is not very far from $1/\langle \Sigma_{2r} \rangle \simeq 0.42$.

One fundamental property of the renormalization group transformation (14) is that the standard deviation of Σ_2 is strictly less than of μ. Hence the transformation is contractive and hence upon the coarse graining, the friction coefficient of the entire system converges toward a δ distribution. Thus to have access to the γ exponent one can consider the asymptotic behaviour assuming a small variance $(b - a) \ll (b - a)/2$.

It is interesting to examine the case of small static friction coefficients:

$$\mu_{ij} = 1 - \varepsilon_{ij}, \quad 0 < \varepsilon_{ij} \ll 1 \tag{15}$$

In this case the value of Σ_2 depends only of the friction coefficients in that diagonal where the product is larger

$$\text{if } b+c > a+d \qquad \Sigma_2 = \frac{b+c}{2} \tag{16a}$$

$$\text{if } b+c < a+d \qquad \Sigma_2 = \frac{a+d}{2} \tag{16b}$$

$$\text{or} \qquad \Sigma_2 = \frac{1}{2}\min(a+d, b+c)$$

The crucial (surprising) result is that although 3 cells have reached the yield limit only 2 diagonally opposite cells dictate the role of Σ_2! Asymptotically $3N^2/4$ inequalities are satisfied but $\Sigma_2(N)$ is given only by N yield limits.

This problem is very similar to finding the ground state of a directed polymer on a hierarchical diamond lattice [9-12]. Let us define this problem.

In a given lattice, we assign to each bond i a random number x_i picked randomly from a given distribution. All path $\mathcal{P}$ that cross the system are given a weight $\|\mathcal{P}\|$, which is the sum of all weights along the path:

$$\|\mathcal{P}\| = \sum_{i\in\mathcal{P}} x_i$$

We search for the path $\mathcal{P}^*$ that minimizes the above norm over the set of all path.

$$\mathcal{P}^* = \frac{1}{L}\min_{\mathcal{P}} \|\mathcal{P}\|$$

The path $\mathcal{P}^*$ is the conformation of a polymer interacting with a quenched random environment at zero temperature. The x_i are the interaction energy of a monomer with the site i. A number of interesting features have been obtained concerning the directed polymer problem. First the conformation $\mathcal{P}^*$ has a self affine geometry with a roughness exponent $\zeta = 2/3$ in two dimension, i.e. transverse fluctuations of the path over length L scales as L^ζ. Furthermore the energy of the polymer for a finite size system scales as

$$\frac{\langle\|\mathcal{P}^*\|\rangle}{L} \sim e_\infty + fL^{-\gamma}, \tag{17}$$

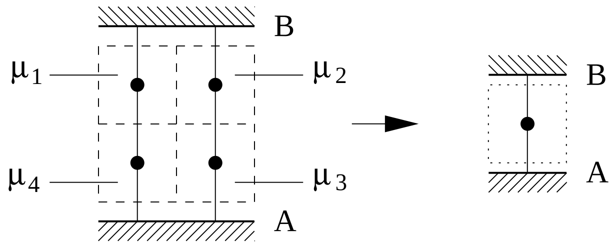

Figure 10: The mapping between directed polymers and our model. The energy between A and B replaced is replaced by a single value calculated from a, b, c and d.

where $\gamma = 2/3$ in two dimension.

Similar scaling property can also observed in three dimension with non trivial exponents (ζ, γ) whose value is not known exactly.

Let us now analyze how this problem can be addressed using a similar renormalization group analysis as previously. The mapping we propose can be seen on Fig. 10.

In a simple 2×2 cell the energy of the polymer $\mathcal{P}^*$ is simply

$$||\mathcal{P}^*|| = \min(a+b, c+d).$$

Thus the energy per monomer $||\mathcal{P}^*||/L$ is rescaled as the μ coefficient in our renormalization group construction for a small disorder. Here Σ_2 behaves as in Eq. 17 with however an exponent γ, which differs from the two dimensional Euclidean value. Previous studies have obtained $\gamma \simeq 1/1.42 \simeq 0.70$.

This justifies the scaling obtained displayed in Fig. 9.

It is well-known that a renormalization group approach gives only approximate value of the critical exponents as compared to Euclidean networks. Thus the key point is not the value of γ. However, identification of universality class or test of relations between critical exponents are robust and can be tested in a renormalized group approach with confidence. Therefore the suggested result is that our perfect plasticity problem can be matched to a directed polymer problem for which the γ exponent should be equal to $\gamma = 2/3$.

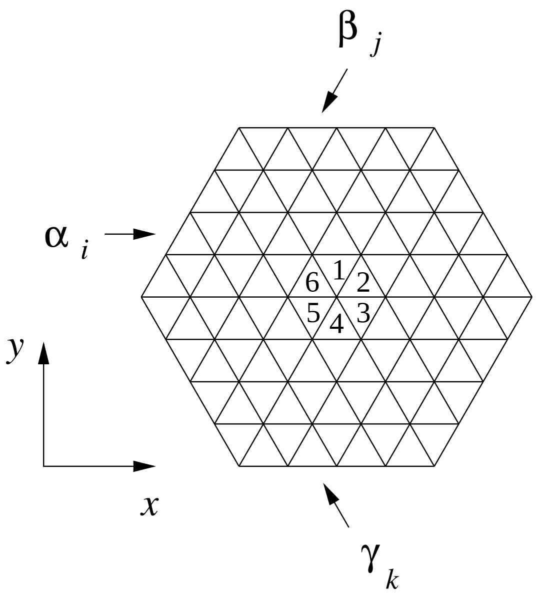

Figure 11: Triangular discretization of the model described in Section 3. Here we have three different row directions and three different constants: α, β and γ. The numbers are for the boundary conditions Eq. (18a) and (18b).

5 Discretization of our model on a triangular lattice

The weakness of the model presented in section 1. is that the number of degrees of freedom is proportional to N, while the number of the yield limits is proportional to N^2. This very small degree of freedom forces Σ_2 to lie close to its lower boundary. One may suspect that the square lattice used the stress discretization is responsible for the large reduction in effective degrees of freedom. Here we show that the type of the lattice is not responsible for this simplicity.

Let us now use a triangular lattice as the base of the discretization. In every site we consider a constant stress tensor and static friction coefficient. The balance equations (2) takes the following form:

$$\text{at } 60^{\text{o}} \text{ angle}: \quad \frac{\sqrt{3}}{2}\sigma_1^{xx} - \frac{1}{2}\sigma_1^{xy} = \frac{\sqrt{3}}{2}\sigma_2^{xx} - \frac{1}{2}\sigma_2^{xy} \quad \text{and} \tag{18a}$$

$$\frac{\sqrt{3}}{2}\sigma_1^{xy} - \frac{1}{2}\sigma_1^{yy} = \frac{\sqrt{3}}{2}\sigma_2^{xy} - \frac{1}{2}\sigma_2^{yy} \tag{18b}$$

$$\text{at } 120^o \text{ angle}: \quad \frac{\sqrt{3}}{2}\sigma_1^{xx} + \frac{1}{2}\sigma_1^{xy} = \frac{\sqrt{3}}{2}\sigma_2^{xx} + \frac{1}{2}\sigma_2^{xy} \quad \text{and} \tag{18c}$$

$$\frac{\sqrt{3}}{2}\sigma_1^{xy} + \frac{1}{2}\sigma_1^{yy} = \frac{\sqrt{3}}{2}\sigma_2^{xy} + \frac{1}{2}\sigma_2^{yy} \tag{18d}$$

$$\text{Horizontally}: \quad \sigma_2^{xy} = \sigma_3^{xy} \quad \text{and} \tag{18e}$$

$$\sigma_2^{yy} = \sigma_3^{yy} \tag{18f}$$

A small calculation shows that in each row (horizontal, at 60^o and at 120^o angle, see Fig. 11) there is a quantity that is constant for all triangles. In horizontal rows this constant has the value

$$\alpha_i = 3\sigma_{ij}^{xx} - \sigma_{ij}^{yy},$$

for all j, where i labels the rows vertically and j the sites from left to right.

By symmetry we can find the same type of constants in the two other rows (Fig. 11). It means that the system is analog to the previous one. We have an invariant for all rows parallel to any principal axis of the lattice. These invariants $\mathcal{O}(N)$ are sufficient to determine the stress in each cell $\mathcal{O}(N^2)$.

6 Presentation of an other formulation using Airy stress potential

From the previous discussion, we are lead to abandon the constancy of the stress over each cell. The easiest way to fulfill the continuity equations (2) in two dimensions is through an *Airy function* or *Airy stress potential.* We recall that this scalar function A, gives the stress tensor field from second derivatives:

$$\sigma_{xx} = \partial_{yy}A(x,y) \tag{19a}$$

$$\sigma_{yy} = \partial_{yy}A(x,y) \tag{19b}$$

$$\sigma_{xy} = -\partial_{xy}A(x,y), \tag{19c}$$

where

$$\partial_i \equiv \frac{\partial A}{\partial x_i} \quad \text{and} \quad \partial_{ij} \equiv \frac{\partial^2 A}{\partial x_i \partial x_j}.$$

For all $A(x,y)$ function the stress continuity equations (2) are automatically fulfilled. For the Coulomb-criteria we need the eigen values of the stress tensor. They are nonlinear functions of the stress components and the problem appears difficult in that general form. As mentioned in the introduction, we expect quite generally that a *weak* disorder will survive at large enough scales. Thus we will assure that μ has a narrow distribution around its mean, and here we will be able to linearize the governing equations. We take the Airy function in the following form

$$A(x,y) = \frac{1}{2}x^2 + \frac{\Sigma_2}{2}y^2 + A_p(x,y), \tag{20}$$

A_p is periodic in both x and y directions. This definition is convenient because we put the leading part into the first two terms and we can keep $A_p(x,y)$ in order of $(b-a)$.

Here we assume that the second derivatives are small $\partial_{ij}A_p \ll 1$. In this case the Coulomb-criteria (1) at the order of $\mathcal{O}(b-a)$ reduces to two linear inequalities.

$$\mu(\Sigma_2 + \partial_{yy}A_p) \;<\; 1 + \partial_{xx}A_p \tag{21a}$$

$$\mu(1 + \partial_{xx}A_p) \;<\; \Sigma_2 + \partial_{yy}A_p \tag{21b}$$

If we specify that $\Sigma_2 > 1$ the second inequality is always fulfilled.

We will solve this system using the Green function (G) of the first (21a) inequality in the equation limit:

$$\partial_{xx}G - \mu\partial_{yy}G = \delta(x)\delta(y) \tag{22}$$

We can assume, that in this hyperbolic equation the y coordinate takes the role of the time and there is a source in the origin. The green function is a wave propagation with $\mu^{-1/2}$ velocity that can be calculated easily using two new variables:

$$\alpha \equiv (x + \sqrt{\mu}y)/\sqrt{2}, \quad \beta \equiv (x - \sqrt{\mu}y)/\sqrt{2}$$

Using these definitions the equation (22) reads in the following form:

$$\partial_\alpha\partial_\beta G(\alpha,\beta) = \delta(\alpha)\delta(\beta) \tag{23}$$

From (23) the Green function can be extracted easily:

$$G(x,y) = H(x + \sqrt{\mu}y)H(x - \sqrt{\mu}y), \tag{24}$$

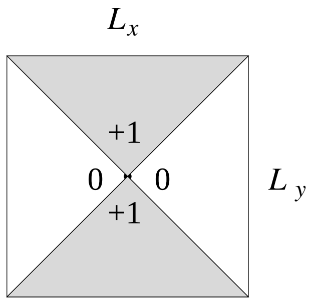

Figure 12: Visualization of the periodic Green function of the equation (25)

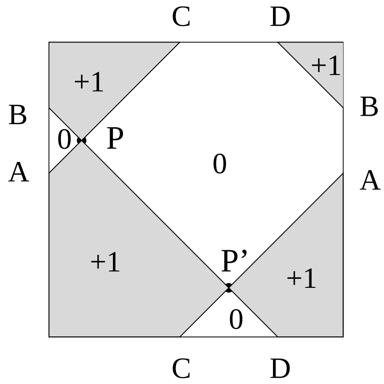

Figure 13: The Green function of the equation (22) in an arbitrary point for special system shape: $L_y/L_x = \sqrt{\mu}$, the same letters denote the same points.

where $H(x)$ denotes the Heaviside function. The problem with this (24) Green function is that it is not periodic that we demand for A_p. In order to solve this problem we will use a periodic Green function:

$$G(x, y) = H(x + \sqrt{\mu}y)H(x - \sqrt{\mu}y) + H(-x - \sqrt{\mu}y)H(-x + \sqrt{\mu}y) \quad (25)$$

A visualization of this periodic Green function can be seen on Fig. 12.

Now we consider the case when we have a finite sample, with $L_y/L_x = \sqrt{\mu}$ and a source at point P (Fig. 13). One can remark that there is a negative source at P'. The distance between P and P' is always $(L_x/2, L_y/2)$. It means that all the points in the system has a pair.

Now let us turn back to our problem. The strength of these sources depends only on the local $\mu(x, y)$. If Σ_2 is less than any μ than every source contributes less than its value, which is permitted by the inequalities. If Σ_2 is larger than a given μ it does not mean automatically that the sample fails because it can be compensated by its pair. This can be done to the point where both of them reach its maximum. Let us denote by μ_1 and μ_2 the pair where we first reach this limit. In this case $1/\Sigma_2 = (\mu_1 + \mu_2)/2$.

Searching for the maximum of Σ_2 is the same problem as looking for the maximum of $(\mu_1 + \mu_2)/2$. In the infinite system it is μ_{max} this gives $\Sigma_2 = 1/\mu_{max}$.

Now we are interested in the size dependence of the Σ_2. The sample is considered to be composed from $N \times N$ points. We have $\mathcal{O}(N^2)$ points so the maximal value of the μ goes as N^{-2} to the maximal value (μ is uniformly distributed). This gives, that the sum $(\mu_1+\mu_2)/2$ goes as N^{-1} to its maximal value (μ_{max}). We can conclude the size dependence of the maximal possible deviatoric stress is the following:

$$\Sigma_2 = \Sigma_\infty + cN^{-1}, \tag{26}$$

where $\Sigma_\infty = 1/\mu_{max}$.

7 Conclusion

In this work we have examined the problem of homogenization of a disordered rigid plastic medium with local yield criterion of Mohr-Coulomb type. The static friction is a random function with a small fluctuating part compared to its mean. We searched for the maximal external load that the material can balance without plastic flow.

In our first model we used a piece-wise constant stress field on the sites. We have developed two algorithms that solve this optimization problem. This based on a hard-core interaction among the yield criteria. It turned out that this model has very few degrees of liberty ($\mathcal{O}(N)$) and gives only trivial exponent $\gamma = 1$ and asymptotic value $\Sigma_\infty = 1/b$, where b is the higher limit of μ. We justified these numerical results theoretically.

We tried also a renormalization group approach of this model based upon the fact that the 2×2 system can be solved exactly. Here the number of the satisfied yield criteria was proportional to $\mathcal{O}(N^2)$ and lead to interesting non-trivial features. A mapping onto a directed polymers in random media was established. We compared the two systems in a limit case and justified this analogy.

The difference behaviour between the first approach and the renormalization group analysis was attributed to the different number of degrees of freedom. After having shown that the lattice chosen was not responsible for this effect, we introduced an another approach. Here we use the Airy function to fulfill automatically the balance equations. We linearized the yield criteria for small variances of μ respect to its mean. This gives a similar optimization problem as the previous model but with a larger degrees of freedom ($\mathcal{O}(N^2)$). Unfortunately, this also increases dramatically the convergence time of our algorithm thus with this method we could only examine very small systems. The results are not good enough to give a conclusive statement on the value of γ.

In the close future we plan to develop a perturbative approach based on the method of characteristics to be applied to the hyperbolic equation obeyed by the Airy potential when the Coulomb-criterion is imposed at each point. We also intended to use a relaxation algorithm instead of the hard-core event driven one, which in related simplex problems revealed to be efficient [3,4].

Appendix A: The expectation value of the free rows

Description of the problem: We have n boxes. We put k stones randomly in the boxes. A box can contain any number of stones. We search the expectation value of the empty boxes $E_k(n)$.

Solution: We throw the stones one by one into the boxes. Let us denote by $\mathcal{P}_k(m)$ the probability that after throwing k stones we have exactly m free boxes. We cat state the following recursive expression for this quantity:

$$\mathcal{P}_k(m) = \frac{n-m}{n}\mathcal{P}_{k-1}(m) + \frac{m+1}{n}\mathcal{P}_{k-1}(m+1).$$

The first term is the probability of the event that after throwing $k-1$ stones we have m free boxes and we throw the kth one into a non empty one. The second term is the probability of the event that after throwing $k-1$ stones we have $m+1$ free boxes and we throw the kth one into an empty one. Only these two events can lead to the required state.

Now we calculate the expectation value of the free boxes after throwing k stones into n boxes.

$$E_k(n) = \sum_{m=0}^{n} m\mathcal{P}_k(m) = \sum_{m=0}^{n} \frac{m(n-m)}{n}\mathcal{P}_{k-1}(m) + \sum_{m=0}^{n} \frac{m(m+1)}{n}\mathcal{P}_{k-1}(m+1)$$

We can change the index of the second sum using the fact that there is a zero after it for $m=0$ and for $m=n$. We take $m' = m+1$.

$$E_k(n) = \sum_{m=0}^{n} m\mathcal{P}_{k-1}(m) - \sum_{m'=0}^{n} \frac{1}{n}\mathcal{P}_{k-1}(m') = \left(1-\frac{1}{n}\right) E_{k-1}(n).$$

The two m^2 terms vanish and only expectation values at step $k-1$ remain. We know that $E_1(n) = n-1$. So we get the solution of the recursion in the following form:

$$E_k(n) = \left(1-\frac{1}{n}\right)^{k-1}(n-1).$$

In particular for $k=n$ we have

$$E_n(n) = (n-1)\left(1-\frac{1}{n}\right)^{n-1} = n\left(1-\frac{1}{n}\right)^{n} \quad \xrightarrow[n\to\infty]{} \quad \frac{n}{e}$$

The latter expression is the one needed in Section 3.

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