Shear band formation as a variational problem

T. Unger & D. E. Wolf
Department of Theoretical Physics, University Duisburg-Essen, Germany
J. Török
Department of Chemical Information Techn., Budapest University of Technology and Economics, Hungary
J. Kertész
Department of Theoretical Physics, Budapest University of Technology and Economics, Hungary

We give a theoretical analysis of recent experiments on localized shear flow, where in a modified Couette cell shear bands were created in the bulk away from the confining walls. We discuss how the shape of the shear bands is related to the cell geometry. First a geometric argument is presented for narrow shear bands which connects the function of their surface position with the shape in the bulk. Assuming a simple dissipation mechanism we show that the principle of minimum dissipation of energy provides a good description of the shape function. Furthermore, we discuss the behavior of shear bands which are detached from the free surface and are entirely covered in the bulk.

1 INTRODUCTION

If large enough shear stress is applied on jammed granular material, it fails to sustain the load and starts flowing. One of the most intriguing instabilities in granular media is manifested by departure from fluid-like deformation, distributed throughout the sample, to a localized deformation that occurs along a rather narrow interface between two essentially unstrained parts. This phenomenon, the so-called shear-banding, has been the subject of many experimental and theoretical studies (Mueth et al. 2000; Lesniewska & Mroz 2001; Tejchman & Gudehus 2001; Herrmann 2001; Kolymbas & Herle 2003; Hartley & Behringer 2003; Tejchman 2004) and still continues to raise intereresting questions.

Many different kind of testers are applied in shear experiments (Schwedes 2003), but here we focus on a modified Couette cell (Fenistein & van Hecke 2003; Fenistein et al. 2004) in which localized shear flow in the bulk away from the confining walls can be studied. The experimental setup consists of a cylindrical container filled with sand up to a certain height The bottom is split into an outer ring rotating with the container wall, and a stationary disk of radius R_s in the center. Thus the outer and inner part of the material are rotated relative to each other, which creates a shear band with cylindrical symmetry: It starts at the perimeter of the stationary bottom disk and extends through the bulk up to the free surface.

On the surface the width of the shear band W and the distance R_c of its center position from the symmetry axis was measured. It was found that R_c gets smaller as the filling height H is increased and that the shear band exhibits a nontrivial curved shape. What determines the shape of the shear band is the subject of our theoretical analysis. The results presented here are based on the work of (Unger et al. 2004).

A remarkable property of the surface position R_c is that it depends only on the two length parameters H and R_s but neither on the particle properties (size, shape, friction, hardness), nor on the shear rate. This is contrary to the width of the band, which is affected by the size and shape of the grains while it is insensitive to changes of the slip radius R_s. This suggests that the shape of the shear band can be studied separately from its width. In the followings the width is neglected and it is assumed that the shear band represents an infinitely thin layer between two blocks of material within which no flow occurs (narrow band approximation).

2 SURFACE - BULK RELATION

The shape of the shear band is described by its radius $r(h)$ as the function of the height h in the bulk ($0 \le h \le H$). One might expect that the profile $r(h)$ is the same as the curve of the surface positions $R_c(H)$ obtained for varying filling heights. However, experimental data shows that the behavior of the two curves

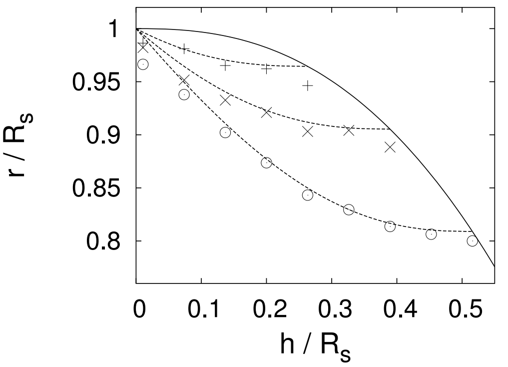

Figure 1. Symbols are experimental data (Fenistein et al. 2004) showing the shear band radius r measured in the bulk at height h for three different filling heights. The solid line is the experimentally found fit curve for the surface positions. The dashed lines are calculated bulk profiles based on the solid curve.

is fundamentally different (see Fig. 1). The radius on the free surface can be well fitted by the following function (Fenistein et al. 2004):

$$R_c(R_s, H) = R_s(1 - (H/R_s)^{5/2}) , \tag{1}$$

while the bulk profiles has opposite curvatures. Furthermore, the bulk radius at a given height h is strongly dependent on the filling height H.

We show that in the narrow band approximation the bulk profile is uniquely determined by the surface position $R_c(R_s, H)$. Let us take a system with total height H and find the position of the shear band r at height $h < H$. The subsystem above h can be regarded as a smaller system with height $(H - h)$ and with slip radius r at the bottom. Pressure and boundary conditions are the same, and the difference in the width is neglected. Of course, the surface radius of the shear band of the subsystem is the same as for the whole system, thus we conclude that

$$R_c(R_s, H) = R_c(r, H - h) . \tag{2}$$

Once the function $R_c(R_s, H)$ is known, this condition allows us to calculate the shape $r(h)$ of the shear band throughout the whole system. Taking the function of R_c given in Eq. 1 the following explicit form of the bulk profile is obtained:

$$h = H - r\left[1 - \frac{R_s}{r}(1 - (H/R_s)^{5/2})\right]^{2/5} . \tag{3}$$

The resulting curves for some filling heights are plotted and compared to experimental data in Fig. 1.

3 VARIATIONAL PRINCIPLE

The fact, that geometry and boundary conditions determine the regions of large shear, is not new. The charm of the experiment discussed here is, however, that nontrivial, and very precisely measured scaling laws were found in this special Couette cell. These scaling laws provide a test for the theoretical description of shear banding.

Very sophisticated continuum models exist to calculate stress and strain rates, and hence also the location of shear bands, with finite element methods (Feise 2000; Tejchman & Gudehus 2001; Kolymbas & Herle 2003; Tejchman 2004). These continuum models are currently discussed controversially, because the relative importance of some of the phenomenological parameters is not clear. The above experiment may serve as a guide.

We show that the well known *principle of least dissipation* (de Groot & Mazur 1969) explains the experimentally observed features. According to this principle the material develops such a steady state flow that it matches the outer constraints while providing the minimum rate of energy dissipation.

The principle of least dissipation is applied in the following way: One has to determine the energy dissipation rate as a functional of the velocity field. For complex equations of motion and stress-strain relationship this may not always be an easy task, but if it can be achieved, the correct velocity field (and hence the location of shear bands) can be obtained by minimizing this functional. This procedure should be equivalent to solving the continuum equations.

We illustrate this procedure in the narrow band approximation, where the velocity field corresponds to a rigid rotation in the outer part of the Couette cell and is zero inside a radius $r(h)$, which marks the position of the shear band. We determine the energy dissipation rate for arbitrary velocity fields of this kind. Then we vary the velocity field to find the solution of minimal energy dissipation. In the narrow band approximation this amounts to varying $r(h)$ with the condition $r(0) = R_s$ while the other boundary at H is free. We compare the position of the shear band calculated in this way with the experiment and find that it fits the experimental results excellently without any free parameter.

The dissipation rate per unit area is provided by the shear stress σ_{tn} in the shear band times the sliding velocity $r(h)\omega$ between the two sides. Integrating this over the variational test surface representing the shear band we get the expression to be minimized:

$$\int_0^H r^2\sqrt{1 + (dr/dh)^2}\,\sigma_{tn} dh = \text{min.} \tag{4}$$

This quantity represents (up to an omitted constant factor) not only the *dissipation rate* but also the *mechanical torque* exerted between the stationary and the rotating part of the system. Therefore the *least dissipation* for this specific geometry is equivalent to the *minimal torque* which gives further justification of this approach: it is plausible that the yielding surface is established where the resistance against the outer

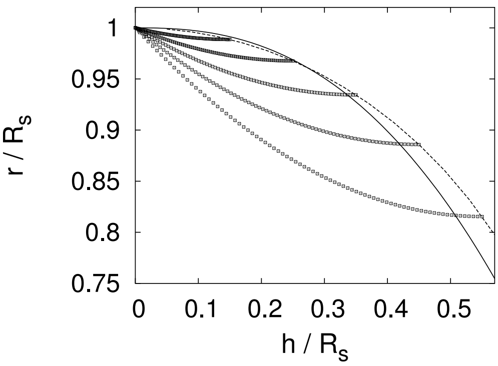

Figure 2. Results obtained from the variational principle. Symbols show bulk profiles, from top to bottom $H/R_s = 0.15, 0.25, 0.35, 0.45, 0.55$ respectively. The two lines denote the surface positions as the function of the total height. The dashed line is provided by our model, the solid line is the experimental fit function.

constraint is the smallest, i.e. where the material is the weakest.

4 SLIDING MODEL

In order to proceed one needs the expression of the shear stress in Eq. (4) and for that we will use a very simple sliding model. The shear stress in the yielding surface is taken similar to the Coulomb friction between two solid bodies: It acts against the sliding direction, its magnitude is proportional to the normal pressure pressing the two sides against each other, but it is independent of the sliding velocity. We assume hydrostatic pressure i.e. pressure proportional to the depth. The Janssen effect (Janssen 1895) where the container wall could carry part of the weight of the material is neglected here. This is justified if H is smaller than the container width. In our dynamical situation, however, we expect that the applicability of the hydrostatic pressure can be extended even for larger filling heights. The shear band (due to many collisions and slip events) acts as a source of vibration. If there was a vertical Janssen-type shear stress at the container wall it would cause a slight creep of the grain-wall contacts under the perturbation effect of the shear band. This inhibits the grains at the wall to keep their original anchoring position and finally they transmit their load to the next particle below rather than to the side wall. Therefore the whole weight is expected to be transmitted to the bottom.

This sliding model leads to the following variational problem:

$$\int_0^H r^2 \sqrt{1 + (dr/dh)^2}\,(H-h)dh = \text{min.}\,. \qquad (5)$$

The solutions which minimize the integral have a simple size-scaling property. If one enlarges the system by a factor λ, i.e. instead of $R_s^{(1)}$ and $H^{(1)}$ one takes $R_s^{(2)} = \lambda R_s^{(1)}$ and $H^{(2)} = \lambda H^{(1)}$ then the profile of the shear band looks the same: the solution will be $r^{(2)}(h) = \lambda r^{(1)}(h/\lambda)$. This is in agreement with the experimental data on R_c which show the same scaling.

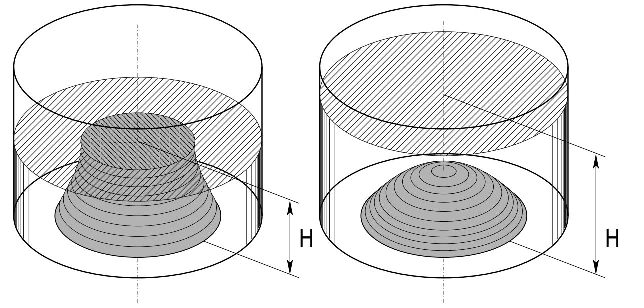

Figure 3. Schematic drawings of open and closed shear bands.

5 NUMERICAL SOLUTION OF THE MODEL

We performed the minimization numerically based on genetic optimization: The function $r(h)$ is discretized and is varied randomly but only the changes are admitted that lower the value of Eq. (5), i.e. reduce the dissipation. During the optimization the noise level is continuously decreased and the final state $r(h)$ is regarded as one local minimum of the variational problem.

Results shown in Fig. 2 reproduce nicely the qualitative behavior found in the experiment: the concave shear bands appear in the bulk and build up a convex confining shape of the surface positions as the filling height is varied.

The curved profiles provided by the principle of minimum dissipation can be interpreted as equilibrium between two effects. Making the shape slimmer at the top reduces the sliding velocity and therefore the energy dissipation per unit area. On the other hand, if the radius is reduced too much at the top it increases the whole surface of the shear band, which counterbalances the first effect.

The quantitative agreement with the experimental fit function (Eq. (1)) is also surprisingly good given the crude assumptions we made and the fact that our model contains no free fit parameters.

6 CLOSED SHAPES

For large filling heights the class of "open" solutions discussed so far is replaced by a new type of solutions (Fig. 3). The shear band, instead of running up to the free surface, closes forming a cupola-like shape, where the material covered by the "closed" band is at rest while the material around and above is rotating.

Several "open" and "closed" profiles can be seen in Fig. 4.b obtained for various values of H. Fig. 4.a shows the upper radius of the shear bands which is characteristic of the "open" solutions but becomes zero for the "closed" ones. For these cupola shapes a more relevant parameter is their heights $h_{\text{top}} \leq H$ in the center, plotted in Fig. 4.c. For "open" profiles h_{top} equals simply the system height.

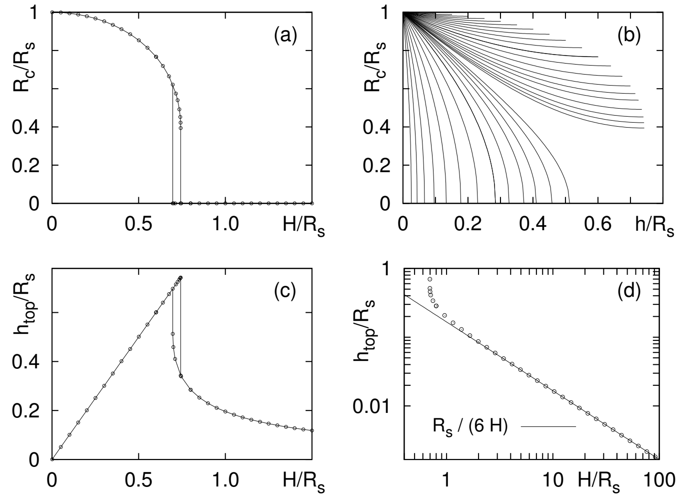

Figure 4. Open and closed shear bands. (a) upper radius, (b) bulk profiles for several filling heights ($0 < H/R_s < 7$), (c,d) height of the shear band h_{top}.

For "closed" bands h_{top} is a monotonically decreasing function of H. By large filling height or alternatively, *by applying additional pressure at the free surface the shear band can be pressed down to the bottom*. To show this we solve the variational problem in the limit $H \gg h_{\text{top}}$, where it is useful to consider the function $h(r)$ instead of $r(h)$. With change of variable we get from Eq. (5)

$$\int_0^{R_s} r^2 \sqrt{1 + (dh/dr)^2}\,(H - h)dr = \text{min.}\,, \qquad (6)$$

where the Euler-Lagrange equation gives

$$h'' r^2 (H - h) + (1 + h'^2)\left[r^2 + (H - h)2rh'\right] = 0\,. \qquad (7)$$

(h' and h'' stand for the first and second derivative of the function $h(r)$.) Retaining only first order terms of h' ($h'^2 \approx 0$) and assuming $H - h \approx H$ we obtain:

$$H\left(h' r^2\right)' = -r^2\,. \qquad (8)$$

This differential equation results in a parabolic profile of the closed shear bands:

$$h(r) = h_{\text{top}} - \frac{1}{6H} r^2\,, \qquad h_{\text{top}} = \frac{1}{6H} R_s^2\,. \qquad (9)$$

Fig. 4.d shows the numerical solution of h_{top} : it is in excellent agreement with the approximate analytical solution.

7 CONCLUSIONS

In this paper we have presented a theoretical analysis of recent experiments on shear band formation. We applied the approximation of narrow bands and a variational principle to describe the shape of the shear bands. The theory provides two kinds of shape depending on the filling height: i) a curved cylinder which ends on the surface and ii) a cupola-like form buried in the bulk. The results are in good agreement with the experiments. Some predictions are also presented here concerning the pressure dependence of the cupola height or the transition between the different forms of the shear bands. It is the task of further experiments to verify or disapprove these predictions.

ACKNOWLEDGMENT

We would like to thank to D. Fenistein and M. van Hecke for useful discussions. Partial support by OTKA T035028 and by the German-Hungarian Cooperation Fund is acknowledged.

REFERENCES

de Groot, S. R. & Mazur, P. 1969. *Non-equilibrium thermodynamics*. Amsterdam: North-Holland Publ.

Feise, H. 2000. *Modellierung des Mechanischen Verhaltens von Schüttgütern*. Ph. D. thesis: TU Braunschweig.

Fenistein, D., van de Meent, J. W., & van Hecke, M. 2004. Universal and wide shear zones in granular bulk flow. *Phys. Rev. Lett.* 92: 094301. cond-mat/0310409.

Fenistein, D. & van Hecke, M. 2003. Kinematics: Wide shear zones in granular bulk flow. *Nature* 425: 256.

Hartley, R. R. & Behringer, R. P. 2003. Logarithmic rate dependence of force networks in sheared granular materials. *Nature* 421: 928.

Herrmann, H. J. 2001. Structures in deformed granular packings. *Granular Matter* 3: 15.

Janssen, H. A. 1895. Versuche über Getreidedruck in Silozellen. *Zeitsch. Ver. Deutsch. Ing.* 39: 1045.

Kolymbas, D. & Herle, I. 2003. Shear and objective stress rates in hypoplasticity. *Int. J. Num. An. Meth. in Geomechanics* 27: 733.

Lesniewska, D. & Mroz, Z. 2001. Study of evolution of shear band systems in sand retained by flexible wall. *Int. J. Num. An. Meth. in Geomechanics* 25: 909.

Mueth, D. M., Debregeas, G. F., Karczmar, G. S., Eng, P. J., Nagel, S. R., & Jaeger, H. M. 2000. Signatures of granular microstructure in dense shear flows. *Nature* 406: 385.

Schwedes, J. 2003. Review on testers for measuring flow properties of bulk solids. *Gran. Mat.* 5: 1.

Tejchman, J. 2004. Comparative fe-studies of shear localizations in granular bodies within a polar and non-local hypoplasticity. *Mech. Research Communications* 31: 341.

Tejchman, J. & Gudehus, G. 2001. Shearing of a narrow granular layer with polar quantities. *Int. J. Num. An. Meth. in Geomechanics* 25: 1.

Unger, T., Török, J., Kertész, J., & Wolf, D. E. 2004. Shear band formation in granular media as a variational problem. *Phys. Rev. Lett.* 92: 214301. cond-mat/0401143.