Multipartite entanglement: the curious case of three qubits

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info: http://indico.kfki.hu/event/315/
Motivation

Multipartite entanglement

- conceptionally richer than bipartite entanglement, paradigmatic new behaviour even for three qubits
- hard to grasp in general
- LOCC/SLOCC-like operative paradigms for entanglement become too complicated, coarse-grainings seem to be enforced
- genuine multipartite correlations can serve as powerful resources
Outline

1 Introduction
   • Motivation and outline
   • One qubit and all that
   • Two-qubit entanglement

2 Three-qubit entanglement
   • Entanglement monogamy
   • Classification
   • Correlation in GHZ experiment
   • Resource

3 Summary
1 Introduction
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   • Correlation in GHZ experiment
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3 Summary
Quantum system

States of a discrete quantum system

- **state vector**: $|\psi\rangle \in \mathcal{H}$ (normalized, $d = \dim \mathcal{H} < \infty$)
- **pure state**: $|\psi\rangle \langle \psi|$  
  we are uncertain in the measurement outcomes,  
  pure state encodes the *probabilities* of those
- **mixed state** (of an ensemble): $\rho = \sum_j p_j |\psi_j\rangle \langle \psi_j|$  
  we are uncertain even in the (pure) state
- **state space** is a **convex** set, extremal points = pure states
- decomposition is not unique
Mixedness of quantum states

Measure of mixedness: entropies

- **von Neumann entropy**: \( S(\rho) = - \text{tr} \rho \ln \rho \)
- **Rényi entropy**: \( S_{R,q}(\rho) = \frac{1}{1-q} \ln \text{tr} \rho^q \)
- **Tsallis entropy**: \( S_{T,q}(\rho) = \frac{1}{1-q} (\text{tr} \rho^q - 1) \)
- **Concurrence-squared**: \( C^2(\rho) = 2S_{T_s}(\rho) = 2(1 - \text{tr} \rho^2) \)

- nonnegative, vanish iff \( \rho \) pure
- **Schur-concave**: entropy = mixedness

concave (Rényi \( 0 < q < 1 \), Tsallis \( 0 < q \)):

increasing w.r.t. forgetting classical information

- Schumacher’s noiseless coding thm:

von Neumann entropy = quantum information content

One qubit

Mixed state
- $\dim \mathcal{H} = 2$, state $\varrho = \frac{1}{2} (I + x\sigma)$, in Bloch sphere $0 \leq \|x\| \leq 1$
- eigenvalues: $\lambda_{\pm} = \frac{1}{2} \left(1 \pm \sqrt{1 - 4 \det \varrho} \right) = \frac{1}{2} \left(1 \pm \|x\| \right)$
- pure state: $\|x\| = 1$, $|\psi\rangle = e^{-i\varphi/2} \cos(\vartheta/2) |0\rangle + e^{i\varphi/2} \sin(\vartheta/2) |0\rangle$
- mixedness: $C^2(\varrho) = 4 \det \varrho = 1 - \|x\|^2$, and $S(\varrho) = S(C(\varrho))$ with $S(x) = h\left(\frac{1}{2} \left(1 + \sqrt{1 - x^2} \right) \right)$, $h(x) = -x \log_2 x - \left(1 - x \right) \log_2(1 - x)$

SL structure
- two-form $\epsilon \in \text{Lin}(\mathcal{H} \to \mathcal{H}^*)$ with antisymmetric $\epsilon_{ii'} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
- action: $\langle \tilde{\psi} | = \epsilon |\psi\rangle$, $|\psi\rangle \mapsto |\tilde{\psi}\rangle = \langle \tilde{\psi} |^*$, $\varrho \mapsto \tilde{\varrho} = (\epsilon \varrho \epsilon^\dagger)^*$
on Bloch vector: $x \mapsto \tilde{x} = -x$ (spin flip)
- transformation: $A^t \epsilon A = (\det A) \epsilon$, so $\text{SL}(2, \mathbb{C})$-invariance
- mixedness: $C^2(\varrho) = 4 \det \varrho = 2 \text{tr} \varrho \tilde{\varrho}$
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- One qubit and all that
- Two-qubit entanglement

Three-qubit entanglement

- Entanglement monogamy
- Classification
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Summary
Bipartite systems and entanglement

Pure States

- $|\psi\rangle \in \mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$
- There are uncorrelated, separable states: $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$.
- Nonclassical: there are also correlated ones, entangled.
  Then measurement on subsystem 1 “causes” the collapse of the state of subsystem 2. (worry of EPR)
- states of subsystems (e.g., $\text{tr}_2 |\psi\rangle\langle\psi|$) are not necessarily pure
- $|\psi\rangle$ is entangled if (and only if) $\text{tr}_2 |\psi\rangle\langle\psi|$ and $\text{tr}_1 |\psi\rangle\langle\psi|$ are mixed
  In this case, “the best possible knowledge of the whole does not involve the best possible knowledge of its parts.” (Schrödinger)
Bipartite systems and entanglement

Pure States
- There are uncorrelated, separable states: $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$.
- Nonclassical: there are also correlated ones, entangled.
- decision of separability is simple: $|\psi\rangle$ is separable iff $\text{tr}_2 |\psi\rangle \langle \psi|$ is pure

Mixed States
- a mixed state is separable if there exists separable decomposition:
  $$\rho = \sum_i p_i (|\psi_{1,i}\rangle \otimes |\psi_{2,i}\rangle) \langle \psi_{1,i} | \otimes \langle \psi_{2,i} |),$$
- classically correlated sources produce states of this kind (Werner)
  can be prepared by Local Operations and Classical Communication
- the others are entangled
- the decomposition is not unique
- decision of separability is difficult
Bipartite systems and entanglement – Pure states

Pure States
- separable: \( |\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \), else it is entangled
- decision of separability is simple: \( |\psi\rangle \) is separable iff \( \text{tr}_2 |\psi\rangle \langle \psi| \) is pure

Measure of entanglement
- von Neumann entropy of entanglement: \( E(|\psi\rangle) = S(\text{tr}_2 |\psi\rangle \langle \psi|) \)
- Rényi entropy of entanglement: \( E_{R,q<1}(|\psi\rangle) = S_{R,q}(\text{tr}_2 |\psi\rangle \langle \psi|) \)
- Tsallis entropy of entanglement: \( E_{T,q}(|\psi\rangle) = S_{T,q}(\text{tr}_2 |\psi\rangle \langle \psi|) \)
- Concurrence of entanglement: \( E_C(|\psi\rangle) = C(\text{tr}_2 |\psi\rangle \langle \psi|) \)
- the mixedness of the subsystem is a good measure of entanglement (entanglement monotone: nonincreasing on average w.r.t. pure LOCC Vidal: isometry-invariant concave function of \( \text{tr}_2 |\psi\rangle \langle \psi| \))
- vanish exactly for separable states
Bipartite systems and entanglement – Mixed states

Mixed States
- separable: \( \rho = \sum_i p_i (|\psi_{1,i}\rangle \otimes |\psi_{2,i}\rangle) (\langle \psi_{1,i} | \otimes \langle \psi_{2,i} |) \), else entangled
- decision of separability is difficult

Measure of entanglement
- the average entanglement of the optimal decomposition
  (convex roof extension of ent. entropy, entanglement of formation)

\[
E(|\psi\rangle) = S(\text{tr}_2 |\psi\rangle\langle \psi|) \quad \sim \quad E^U(\rho) = \min_{\rho=\sum_i p_i |\psi_i\rangle\langle \psi_i|} \sum_i p_i E(|\psi_i\rangle)
\]

is a good measure of entanglement (entanglement monotone)
- similarly, Rényi, Tsallis entanglement of formation \( E_{R,q<1}^U, E_{T,q}^U \),
  Concurrence of formation \( E_C^U \)
- vanish exactly for separable states
Two qubits

Pure States

- \( \dim \mathcal{H}_1 = \dim \mathcal{H}_2 = 2 \), state: \( |\psi\rangle = \sum_{i,j=0}^{1} \psi^{ij} |ij\rangle \in \mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2 \)
- \( E_C(|\psi\rangle) = C(\text{tr}_2 |\psi\rangle \langle \psi|) \) and \( E(|\psi\rangle) = S(E_C(|\psi\rangle)) \)
- we had: \( E^2_C(\psi) = 4 \det(\text{tr}_2 |\psi\rangle \langle \psi|) \)
  - easy: \( E_C(\psi) = 2 |\det \psi| = 2 |\psi^{00}\psi^{11} - \psi^{01}\psi^{10}| \)
    \[
    = |\langle \tilde{\psi}|\psi\rangle| = \epsilon_{ii'}\epsilon_{jj'}\psi^{ij}\psi^{i'j'} = \sqrt[2]{\psi^i\psi^j},
    \]
    \[
    \text{(with the two-qubit spin-flip} \langle \tilde{\psi}| = \epsilon \otimes \epsilon |\psi\rangle)\]
- hard: \( E^U_C(\varrho) = (\lambda_1^\dagger - \lambda_2^\dagger - \lambda_3^\dagger - \lambda_4^\dagger)^+ \), and \( E^U(\varrho) = S(E^U_C(\varrho)) \),
  where \( \lambda_i^\dagger \)'s are the decreasingly ordered eigenvalues of the positive matrix \( \sqrt{\sqrt{\varrho^0\varrho^0}\sqrt{\varrho}} \), written with the spin-flip \( \tilde{\varrho} = (\epsilon \otimes \epsilon \varrho \epsilon^{\dagger} \otimes \epsilon^{\dagger})^* \).
  This is called \textbf{Wootters' concurrence}. \( (\text{SL}(2, \mathbb{C}) \times 2\text{-invariant}) \)

Wootters, PRL 80, 2245 (1998)
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Two-qubit entanglement inside the three-qubit system

Entanglement monogamy

- \( \dim \mathcal{H}_1 = \dim \mathcal{H}_2 = \dim \mathcal{H}_3 = 2 \),
- state: \( |\psi\rangle = \sum_{i,j,k=0}^{1} \psi_{ijk} |ijk\rangle \in \mathcal{H}_{123} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \)
- states of subsystems: \( \varrho_{bc} = \text{tr}_a |\psi\rangle \langle \psi| \) and \( \varrho_a = \text{tr}_{bc} |\psi\rangle \langle \psi| \)
- restriction on bipartite entanglement (monogamy, CKW inequality):
  \[
  E_C^U(\varrho_{ab})^2 + E_C^U(\varrho_{ac})^2 \leq C^2(\varrho_a)
  \]

GHZ: \( |\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \):
\[
0 + 0 < 1
\]
W: \( |\psi_W\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle) \):
\[
4/9 + 4/9 = 8/9
\]
- there’s no such restriction for correlation in classical systems
- moreover, holds for \( n \) qubits \( \sum_{b \neq a} E_C^U(\varrho_{ab})^2 \leq C^2(\varrho_a) \)
- also, entanglement restricts classical correlation, and vice versa

Coffman, Kundu, Wootters, PRL 61 052306 (2000)
Osborne, Verstraete, PRL 96 220503 (2006)
Two-qubit entanglement inside the three-qubit system

Distributed entanglement

- append the monogamy inequality with the difference:

\[ E_C^U(\rho_{ab})^2 + E_C^U(\rho_{ac})^2 + \tau(\psi) = C^2(\rho_a) \]

- three-tangle: \( \tau(\psi) = 4|\text{Det } \psi| \), with Cayley’s hyperdeterminant

\[
\text{Det } \psi = \frac{1}{2} \varepsilon_{ii'}\varepsilon_{jj'}\varepsilon_{kk'}\varepsilon_{ll'}\varepsilon_{mm'}\varepsilon_{nn'} \psi^{ikl} \psi^{jkl'} \psi^{imn} \psi^{jn'n'} = -\frac{1}{2} \psi_0^2 \psi_1^2 \psi_2^2
\]

\[
= \psi_{000}^2 \psi_{111}^2 + \psi_{110}^2 \psi_{001}^2 + \psi_{101}^2 \psi_{010}^2 + \psi_{011}^2 \psi_{100}^2 + \psi_{110}^2 \psi_{001}^2 + \psi_{101}^2 \psi_{010}^2 + \psi_{011}^2 \psi_{100}^2 + \psi_{110}^2 \psi_{001}^2 + \psi_{101}^2 \psi_{010}^2 + \psi_{011}^2 \psi_{100}^2
\]

\[
- 2 (\psi_{000}^2 \psi_{111}^2 \psi_{110}^2 + \psi_{000}^2 \psi_{111}^2 \psi_{001}^2 + \psi_{110}^2 \psi_{001}^2 \psi_{111}^2 + \psi_{110}^2 \psi_{001}^2 \psi_{110}^2 + \psi_{101}^2 \psi_{010}^2 \psi_{111}^2 + \psi_{101}^2 \psi_{010}^2 \psi_{101}^2 + \psi_{011}^2 \psi_{100}^2 \psi_{110}^2 + \psi_{011}^2 \psi_{100}^2 \psi_{101}^2)
\]

\[
+ 4 (\psi_{000}^2 \psi_{111}^2 \psi_{101}^2 \psi_{011}^2 + \psi_{111}^2 \psi_{001}^2 \psi_{100}^2 \psi_{100}^2)
\]

- \( \text{SL}(2, \mathbb{C})^3 \)- and permutation-invariant, entanglement monotone

Coffman, Kundu, Wootters, PRL 61 052306 (2000)
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3 Summary
Some ways of classification

- **Local Operation and Classical Communication (LOCC)**, $\rho \cong_{\text{LOCC}} \omega$: they can be converted into each other by means of LOCC.

- **LOCC classification for pure states**: $|\psi\rangle \cong_{\text{LOCC}} |\varphi\rangle$ iff $|\psi\rangle \cong_{\text{LU}} |\varphi\rangle$: Local Unitary equivalent, $|\varphi\rangle = U_1 \otimes \cdots \otimes U_n |\psi\rangle$, $U_i \in U(\mathcal{H}_i)$

- **Stochastic LOCC (SLOCC)**, $\rho \cong_{\text{SLOCC}} \omega$: they can be converted into each other by means of LOCC with nonzero probability of success.

- **SLOCC classification for pure states**: $|\psi\rangle \cong_{\text{SLOCC}} |\varphi\rangle$ iff $|\psi\rangle \cong_{\text{LGL}} |\varphi\rangle$: Local General Linear equiv., $|\varphi\rangle = \frac{1}{p_{\text{succ}}} G_1 \otimes \cdots \otimes G_n |\psi\rangle$, $G_i \in \text{GL}(\mathcal{H}_i)$

- **Partial Separability (PS)**, $\rho \cong_{\text{PS}} \omega$: see in my next talk.

- **PS classification for pure states**: $|\psi\rangle \cong_{\text{PS}} |\varphi\rangle$ iff the finest way of decomposition to tensor product form are the same.
LOCC classification of two- and three-qubit pure states

can be given by LU canonical form:

- **two qubits**: Schmidt decomposition \( |\psi\rangle = \sqrt{\eta_0} |00\rangle + \sqrt{\eta_1} |11\rangle \), with \( \eta_0 \geq \eta_1 \geq 0 \), (normalized: 1-parameter set of LOCC classes)
- **LOCC conversion (Nielsen)**: \( |\psi\rangle \mapsto |\psi'\rangle \) iff \( \eta_0 \leq \eta'_0 \)
- **three qubits**: canonical decomp. \( |\psi\rangle = \sum_i \sqrt{\eta_i} |\varphi_{1,i}\rangle \otimes |\varphi_{2,i}\rangle \otimes |\varphi_{3,i}\rangle \), not locally orthogonal vectors, border rank problem...
- **three qubits**: generalized Schmidt decomposition
  \[
  |\psi\rangle = \sqrt{\eta_0} |000\rangle + \sqrt{\eta_1} e^{i\phi} |100\rangle + \sqrt{\eta_2} |101\rangle + \sqrt{\eta_3} |110\rangle + \sqrt{\eta_4} |111\rangle,
  \]
  \( 0 \leq \eta_i, \ 0 \leq \phi \leq \pi \), 5-parameter set of LOCC classes

can also be given by a sufficient set of LU invariants

even max. entangled state sets (w.r.t. LOCC) show structure too involved

SLOCC classification of two- and three-qubit pure states

- **Two qubits**: 2 classes, representative elements
  - Entangled: \( |\psi_{\text{ent}}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \)
  - Separable: \( |\psi_{\text{sep}}\rangle = |00\rangle \)
- **Local selective measurements**: entangled \( \rightarrow \) separable
- **Three qubits**: 6 classes, representative elements
  - GHZ: \( |\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \)
  - W: \( |\psi_{\text{W}}\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle) \)
  - 1|23-biseparable: \( |\psi_{1|23}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |011\rangle) \ldots \)
  - Fully separable: \( |\psi_{1|2|3}\rangle = |000\rangle \)
- **Principles**: local rank and minimal number of product terms in decomposition are invariant w.r.t. SLOCC
- **GHZ class**: \( \tau(\psi) > 0 \)
- **Generalizations**: two qubits with a qutrit or qu4it: discrete classes
- Four qubits: nine families of continuously parametrized classes
SLOCC classification of two- and three-qubit pure states

- **two qubits:** 2 classes, representative elements
  - entangled: $|\psi_{\text{ent}}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
  - separable: $|\psi_{\text{sep}}\rangle = |00\rangle$
- **local selective measurements:** entangled $\rightarrow$ separable
- **three qubits:** 6 classes, local selective measurements:

  ![Diagram showing SLOCC classification with W and GHZ classes and their local ranks](image)

- **GHZ class:** $\tau(\psi) > 0$
- **generalizations:** two qubits with a qutrit or qu4it: discrete classes
  - four qubits: nine families of continuously parametrized classes

Dür, Vidal, Cirac, PRA 62062314 (2000)

Szilárd Szalay (MTA Wigner FK, SZFI)
SLOCC classification of two- and three-qubit pure states

- **two qubits**: 2 classes, representative elements
  - **entangled**: \( |\psi_{\text{ent}}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \)
  - **separable**: \( |\psi_{\text{sep}}\rangle = |00\rangle \)
- **local selective measurements**: entangled \( \rightarrow \) separable
- **separable two-qubit subsystems (only) in the GHZ class**

\[
|\psi_W\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle) \quad |\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)
\]

- **generalizations**: two qubits with a qutrit or qu4it: discrete classes
  - four qubits: nine families of continuously parametrized classes
+1: FTS classification of three-qubit pure states

- $H_{123} \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathcal{M}(\mathcal{J}) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathcal{J} \oplus \mathcal{J}$ Freudenthal Triple System over the cubic Jordan algebra $\mathcal{J} \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$

- Each element has an LGL-invariant “Freudenthal rank”, given by the vanishing of the $SL(2, \mathbb{C})$-covariants

\begin{align*}
q(\psi) &= \psi \psi \psi \psi \\
T(\psi) &= \psi \psi \psi \psi = \psi \psi \psi \psi = \psi \psi \psi \psi \\
\gamma_1(\psi) &= \psi \psi \psi \\
\gamma_2(\psi) &= \psi \psi \psi \\
\gamma_3(\psi) &= \psi \psi \psi
\end{align*}

- States are SLOCC equivalents iff Freudenthal ranks are the same, 4: GHZ, 3: W, 2: biseparable, 1: fully separable

- Classification: same as SLOCC, but different hierarchy

- Generalization: only for some special number of special dimensions

Borsten, Dahanayake, Duff, Rubens, Ebrahim, PRA 80 032326 (2009)
+1: FTS classification of three-qubit pure states

\[ \mathcal{H}_{123} \cong \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \cong M(\mathcal{J}) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathcal{J} \oplus \mathcal{J} \]

Freudenthal Triple System over the cubic Jordan algebra \( \mathcal{J} \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \)

\[ \text{GHZ} \leftrightarrow \text{FTS rank} = 4 \]
\[ \text{W} \leftrightarrow \text{FTS rank} = 3 \]
\[ \leftrightarrow \text{FTS rank} = 2 \]
\[ \leftrightarrow \text{FTS rank} = 1 \]

- classification: same as SLOCC, but different hierarchy
- generalization: only for some special number of special dimensions

Borsten, Dahanayake, Duff, Rubens, Ebrahim, PRA 80 032326 (2009)
PS classification of two- and three-qubit pure states

- **Two qubits**: 2 classes, representative elements
  - Entangled: $|\psi_{12}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
  - Separable: $|\psi_{1|2}\rangle = |00\rangle$
- **Local selective measurements**: entangled $\rightarrow$ separable
- **Three qubits**: 5 classes, representative elements
  - Fully entangled: $|\psi_{123}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
  - 1|23-biseparable: $|\psi_{1|23}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |011\rangle)$...
  - Fully separable: $|\psi_{1|2|3}\rangle = |000\rangle$

- Can be given for arbitrary number of arbitrary subsystems, discrete
PS classification of two- and three-qubit pure states

- **two qubits**: 2 classes, representative elements
  - entangled: \( |\psi_{12}\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \)
  - separable: \( |\psi_{1|2}\rangle = |00\rangle \)
- **three qubits**: 5 classes, local selective measurements:
  - can be given for arbitrary number of arbitrary subsystems, discrete
Classifications of two- and three-qubit mixed states

I’m pretty sure that we won’t have time for this one.

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GHZ experiment

- three spin-$\frac{1}{2}$ particles, three detectors with two settings ($\{0, 1\}$) each, with two outcomes ($\{0, 1\}$) each

- settings: restricted to odd number of 0s

- outcomes: $\{011, 101, 110\} \mapsto \{011, 101, 110, 000\}$ (odd number of 0) 
  $\{000\} \mapsto \{100, 010, 001, 111\}$ (even number of 0)

- EPR-like argument: outcomes should be stored in LHVs, but LHVs cannot be given. (Local Hidden Variable)

- quantum mechanical description: state: $|\psi_{\text{GHZ}}^{\prime}\rangle = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$, simultaneous eigenvector of observables: $\sigma_x \otimes \sigma_y \otimes \sigma_y$, $\sigma_y \otimes \sigma_x \otimes \sigma_y$, $\sigma_y \otimes \sigma_y \otimes \sigma_x$, $\sigma_x \otimes \sigma_x \otimes \sigma_x$, with eigenvalues 1, 1, 1, $-1$, respectively.

- not only statistical, but absolute contradiction with LHVM!

Mermin, AmJPhys 58, 731 (1990)
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3 Summary
Classical nonlocal games

Cooperative games of incomplete information

- players (Alice, Béla, . . .) with prearranged collective strategy,
- referee, poses questions \( q \in Q, r \in R, . . . \) (with known prob. \( p \))
- players answer \( a \in A, b \in B, . . . \) (without communication and without knowing the others’ questions)
- referee evaluates (predicate \( V \) on \( A \times B \times \cdots \times Q \times R \times \cdots \))
- deterministic strategy: functions \( a : Q \rightarrow A, b : R \rightarrow B, . . . \)
- classical value of nonlocal game (best winning probability):
  \[
  \omega_c = \max_{a(q), b(r), . . .} \sum_{q, r, . . .} p(q, r, . . .) V(a(q), b(r), . . . | q, r, . . .)
  \]
  (probabilistic strategies: convex combinations of deterministic ones)

Quantum nonlocal games

Cooperative games of incomplete information + shared entanglement

- same situation as before, but entangled multiprätite sys. shared, $|\psi\rangle$
- players have POVMs for each question (strategy),
  Alice: $\forall q \in Q : X_q = \{ X^a_q \mid \sum_{a \in A} X^a_q = I \}$,
  Béla: $\forall r \in R : Y_r = \{ Y^b_r \mid \sum_{b \in B} Y^b_r = I \}$, ...
- they perform the measurement corresponding to the questions
  and answer the outcomes
- quantum value of nonlocal game ($\sim$best winning probability):
  $$\omega_q = \sup \sum_{p(q, r, \ldots)} \sum_{q, r, \ldots} \langle \psi | X^a_q \otimes Y^b_r \otimes \ldots | \psi \rangle V(a, b, \ldots | q, r, \ldots)$$

Quantum nonlocal game of two players

CHSH game

- players: Alice, Bob; questions: \{0, 1\}; answers: \{0, 1\}
- game: \( p \) uniform, \( V(a, b|q, r) = 1 \) if \( a \oplus b = q \land r \), otherwise 0
  that is, 11 \( \mapsto \) odd 1s, 00, 01, 10 \( \mapsto \) even 1s,
- classical value: \( \omega_c = \frac{3}{4} = 0.75 \)

contradictory system:
- \( a(0) \oplus b(0) = 0 \)
- \( a(0) \oplus b(1) = 0 \)
- \( a(1) \oplus b(0) = 0 \)
- \( a(1) \oplus b(1) = 1 \)
best they can do: say always 0
Quantum nonlocal game of two players

CHSH game

- players: Alice, Bob; questions: \{0, 1\}; answers: \{0, 1\}
- game: \( p \) uniform, \( V(a, b|q, r) = 1 \) if \( a \oplus b = q \land r \), otherwise 0
  that is, 11 \( \mapsto \) odd 1s, 00, 01, 10 \( \mapsto \) even 1s,
- classical value: \( \omega_c = 3/4 = 0.75 \)
- quantum value: \( \omega_q = \cos^2(\pi/8) = \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.85 \)

share \( |\psi_B\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \) states,
\( X_0 \): proj. spin meas. along \((0, 0, 1)\) direction (up: 0; down: 1)
\( X_1 \): proj. spin meas. along \((1, 0, 0)\) direction
\( Y_0 \): proj. spin meas. along \((1, 0, 1)/\sqrt{2}\) direction
\( Y_1 \): proj. spin meas. along \((1, 0, -1)/\sqrt{2}\) direction
(Tsirelson bound: optimal for shared qubits)

Quantum nonlocal game of three players

Three-qubit game

- players: Alice, Bob, Charlie; questions: \{0, 1\}; answers: \{0, 1\}
- questions: restricted to odd number of 0s
- game: \( p \) uniform,
  \[
  V(a, b, c|q, r, s) = 1 \text{ if } a \oplus b \oplus c = q \lor r \lor s, \text{ otherwise } 0,
  \]
  that is, 000 \(\mapsto\) even 1s, 011, \ldots \(\mapsto\) odd 1s,
- classical value: \( \omega_c = 3/4 \)
- contradictory system:
  \[
  \begin{align*}
  a(0) \oplus b(0) \oplus c(0) &= 0 \\
  a(0) \oplus b(1) \oplus c(1) &= 1 \\
  a(1) \oplus b(0) \oplus c(1) &= 1 \\
  a(1) \oplus b(1) \oplus c(0) &= 1
  \end{align*}
  \]
  best they can do: say always 1
Quantum nonlocal game of three players

Three-qubit game

- players: Alice, Bob, Charlie; questions: \{0, 1\}; answers: \{0, 1\}
- questions: restricted to odd number of 0s
- game: \( p \) uniform,
  \[ V(a, b, c|q, r, s) = 1 \text{ if } a \oplus b \oplus c = q \lor r \lor s, \text{ otherwise } 0, \]
  that is, 000 \( \mapsto \) even 1s, 011, \ldots \( \mapsto \) odd 1s,
- classical value: \( \omega_c = 3/4 \)
- quantum value: \( \omega_q = 1! \)

share \[ |\psi''_{\text{GHZ}}\rangle = \frac{1}{2}(|000\rangle - |011\rangle - |101\rangle - |110\rangle) = (PH)^{\otimes 3} |\psi_{\text{GHZ}}\rangle \]

\( X_0, Y_0 \): proj. spin-z meas. (|0\rangle, |1\rangle basis)
\( X_1, Y_1 \): proj. spin-x meas. ((|0\rangle + |1\rangle)/\sqrt{2}, (|0\rangle - |1\rangle)/\sqrt{2})

\( qrs = 000 \): even 1s: always win
\( qrs = 011 \): odd 1s: always win,
check: \( I \otimes H \otimes H |\psi''_{\text{GHZ}}\rangle = \frac{1}{2}(|001\rangle + |010\rangle - |100\rangle + |110\rangle) \)
Quantum nonlocal game of three players

Three-qubit game with other kinds of entanglement

- **game**: 000 $\mapsto$ even 1s, 011, ... $\mapsto$ odd 1s,
- **classical value**: $\omega_c = \frac{3}{4}$
- **quantum value** with shared GHZ-class states (FTS rank-4): $\omega_{GHZ} = 1$
- from rank conditions: for shared $W$-class states (FTS rank-3), $\omega_W < 1$
- winning GHZ strategy for $W$ states: $\frac{7}{8} = 0.875 \leq \omega_W$
- for shared biesparable states (FTS rank-2):
  - Tsirelson-type construction, $\omega_{bisep} \leq \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.85$
  - equivalent to CHSH game, so this bound is attained
- **uncorrelated** states (FTS rank-1) is like having nothing, $\omega_{sep} = \omega_c$

$$\omega_c = \omega_{sep} = \frac{3}{4} < \omega_{bisep} = \frac{1}{2} + \frac{1}{2\sqrt{2}} < \frac{7}{8} \leq \omega_W < 1 = \omega_{GHZ}$$
Summary

Multipartite entanglement

- conceptionally richer than bipartite entanglement
- paradigmatic new behaviour even for three qubits
  - monogamy: restriction on entanglements inside subsystems
  - classification issues: LOCC/SLOCC, FTS, PS
  - GHZ correlations: absolute contradiction with LHVM
  - quantum nonlocal games: resources of different values
    (not LOCC/SLOCC situation, but still local, operative approach)
- hard to grasp in general
- LOCC/SLOCC-like operative paradigm for entanglement becomes too complicated, coarse-grainings seem to be enforced
Thank you for your attention!

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