POWER LAW BEHAVIOR IN DYNAMIC NUMERICAL MODELS OF STOCK MARKET PRICES

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1. Introduction

Fluctuations in stock market prices and foreign exchange rates are important not only for investors but also for everyone, because today the world's economies are deeply interrelated, and a crash in one country might cause a global depression or even worldwide panic. A serious problem is that there is no established theory to discriminate whether a given fluctuation is healthy or dangerous. It is an urgent task for scientists to elucidate the mathematical nature of price fluctuations so that we can avoid panics in which most people lose rationality.

From the mathematical point of view, power spectral analysis of such fluctuations always gives roughly an inverse square power spectrum with respect to frequency. This type of power spectrum shows that price changes at each time interval can be considered as an independent stochastic event, which is essential for the market to be a fair gamble to every investor. An interesting and basic open problem is the distribution of price changes. It is widely recognized that price fluctuations obviously include large events more frequently than normal stochastic processes based on the Gaussian process [3].

In the 1960's Mandelbrot pointed out the underlying relation between the price changes and the Levy stable distribution [1]. Recently, Mantegna and Stanley have analyzed a huge set of price change data using methods from statistical mechanics and clarified that the distribution function of price changes has power tails with characteristic exponent 1.4, consistent with the stable distribution [2]. The aim of this article is to provide a theoretical insight into the appearance of these power law distributions.

In the next section we discuss one of the classical problems of economics, namely the balance of supply and demand, from a statistical physics viewpoint, and we show that the equilibrium state can be considered as a critical state near a phase transition. In the third section we first discuss that the basic transactions of buying and selling in a market are highly nonlinear and irreversible processes. Then we introduce the threshold model of dealers which simplifies the transaction in a market. Although the time evolution rule of the threshold model is deterministic, the resulting simulated time evolution is well approximated by a stochastic evolution rule, as shown in the fourth section. The last section is devoted to the discussion of future problems.

2. The balance of demand and supply

In basic economics we assume that a price is determined by the crossing point of the demand and supply curves. When the quantities of demand and supply are both infinitely large, this balance is stable and the price is also stable. However, if these quantities are limited, then we have to take into account the effect of fluctuations around the mean values. Let $D(t)$ and $S(t)$ be the quantities of demand and supply in a unit time interval at time t with the price $p(t)$, then the equilibrium condition is given by

$$\langle D(t)\rangle = \langle S(t)\rangle. \tag{1}$$

Let us investigate the stochastic fluctuations about this mean value. Denoting the fluctuations for demand and supply by $d(t)$ and $s(t)$, we consider the case that $d(t)$ and $s(t)$ are independent white noises with zero means. The total amount of demand minus supply is

$$I(t) = \sum_{t'=0}^{t} \{D(t') - S(t')\}. \tag{2}$$

It is trivial that $D(t') - S(t') = d(t') - s(t')$ is also a white noise so $I(t)$ follows a Brownian motion. Namely, even if the mean demand and supply are in equilibrium, we have Brownian fluctuations, which are characterized by the inverse square power spectrum.

This Brownian fluctuation can be regarded as a kind of critical fluctuation accompanied by a phase transition. In the basic model of demand and supply, there are obviously two phases, the excess demand phase and the excess supply phase. The control parameter is the market price p. For p smaller than the balanced price, we have the excess demand phase, in which $I(t)$ increases infinitely, while for larger p, $I(t)$ decreases monotonically falling into the excess supply state. At the critical value of p, the mean value of $I(t)$ is zero, but it fluctuates according to the above Brownian motion. The ideal Brownian motion is not stationary, so the deviation from $I(t) = 0$ can have any magnitude and any duration in time. This means that the critical point is not stable, due to the intrinsic fluctuations of demand and supply.

If the market is efficient enough, the equilibrium market price is shifted to make $I(t)$ vanish, i.e. the market has a mechanism to tune the control parameter automatically to the critical point. In this sense, the market can be viewed as a kind of self-organized critical system. When demand is larger than supply, the market price should increase, so a simple assumption is that the price, $p(t)$ will roughly be proportional to $I(t)$. A natural consequence of this is that the market price fluctuates following a Brownian motion whenever the market capacity is finite and the price is sensitive to the change of demand and supply, i.e. the price elasticity is small, in the terminology of economics.

3. The deterministic threshold model

As we have discussed in the preceding section, the market price cannot be stable on small markets. Here, we first discuss the extreme limit of the transaction of a single stock between two dealers.

In general, every dealer in a market has two prices for each brand in mind, selling and buying prices. A dealer's selling price is the threshold price above which he wants to sell. Similarly, if the market price is lower than this buying price, he buys the stock immediately. Let us denote the selling and buying prices for dealer 1 as S_1 and B_1, respectively, and those for dealer 2 as S_2 and B_2. S_i is always larger than B_i, otherwise the dealer could sell a stock to himself, which is completely absurd.

A transaction can take place if B_1 is higher than S_2 (or B_2 is higher than S_1). Then dealer 1 buys a stock from dealer 2 at a price between S_2 and B_1 (or vice versa). This transaction is highly nonlinear and irreversible in the following sense. Let us consider the situation when dealer 1 wants to buy a stock, so he gradually raises his prices in his mind while dealer 2 keeps his prices unchanged. As long as B_1 is smaller than S_2, no transaction will take place. Just at $B_1 = S_2$, there suddenly occurs a transaction, which is responsible for the highly nonlinear threshold dynamics. At that time the 4 prices satisfy the following relation:

$$B_1 < S_1 = B_2 < S_2. \tag{3}$$

This process is irreversible because the inverse transaction requires $B_1 = S_2$, but this equality can never be satisfied.

As known from chaos theory, nonlinearity and irreversibility are the sources of complex dynamics. Actually, one of the authors (H.T.) and coworkers demonstrated that a deterministic model of dealers with irreversible threshold dynamics shows a chaotic time evolution with the maximum Lyapunov exponent being close to zero [7].

Let us introduce the revised dealer model [4]. For simplicity we consider a stock market with N dealers trading in only one brand. The selling price for the i-th dealer, S_i, is given by $B_i + L$, where L is a positive constant. A transaction will take place whenever the following condition is satisfied:

$$\max\{B_i\} - \min\{B_i\} \geq L, \tag{4}$$

where max{...} and min{...} denote the maximum and minimum values. We assume that a transaction occurs between the two dealers who propose the maximum buying price and the minimum selling price.

The market price, $P(t)$, is defined by the mean value of $\max\{B_i\}$ and $\min\{B_i\} + L$ when a transaction occurs. When no trade conditions are satisfied, the market price is kept constant.

At every time step, each dealer updates his price according to the following deterministic rule:

$$B_i(t+1) = B_i(t) + a_i(t) + c\{P(t) - P(t')\}, \tag{5}$$

where $a_i(t)$ denotes the i-th dealer's expectation of bid price at time step t and t' denotes the time when the last transaction occurred and c is a constant coefficient showing the response to the market price changes.

The dynamics of $a_i(t)$ characterizes the behavior of the i-th dealer. When $a_i(t)$ is positive, the i-th dealer increases the price in his mind, meaning that he wants to sell stocks, and for negative $a_i(t)$, the dealer wants to buy stocks. We assume a limit situation where all dealers have small amounts of property and each dealer changes his position from buyer to seller after he buys a stock and vice versa. This rule can be implemented by adding a rule that $a_i(t)$ changes its sign after the i-th dealer was involved in a transaction. The absolute value of $a_i(t)$ which characterizes the dealer's hastiness is given initially by a random number and is kept constant.

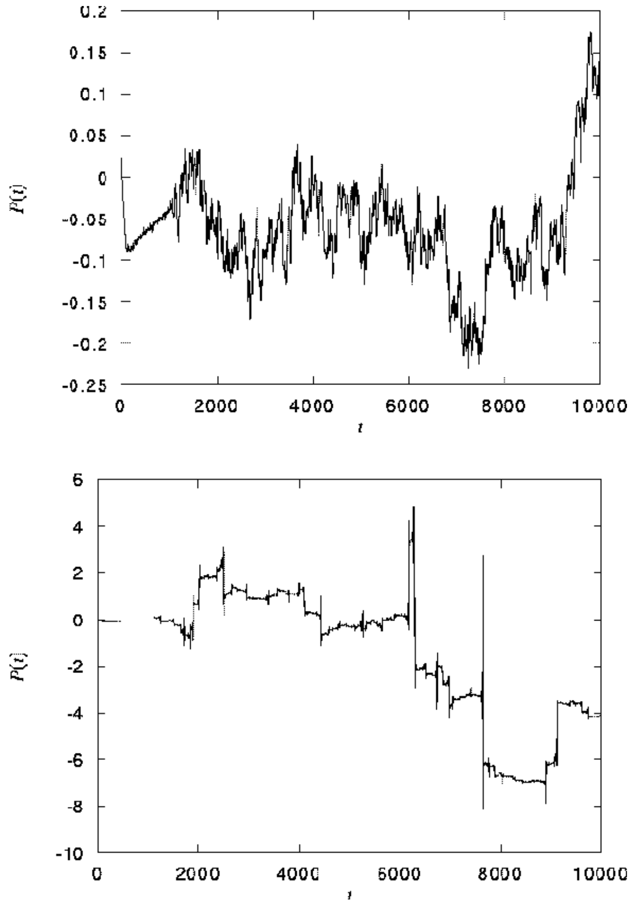

Figure 1. Examples of temporal fluctuations of market price

Examples of the time evolution are shown in Fig.1. The number of dealers in the simulation is $N = 100$ and $\{a_i(0)\}$ are set randomly in the interval $[-1, 1]$. The initial values $\{B_i(0)\}$ are not sensitive to the price change statistics after some number of time steps, for example, 2000. We always have fluctuations that are characterized by the inverse square power spectrum for $c \neq 0$.

Following the real market analysis by Mantegna and Stanley [2] we observe market price changes,

$$\Delta P(t) = P(t) - P(t'), \tag{6}$$

and estimate the probability density function (PDF for short) of $\Delta P(t)$. For

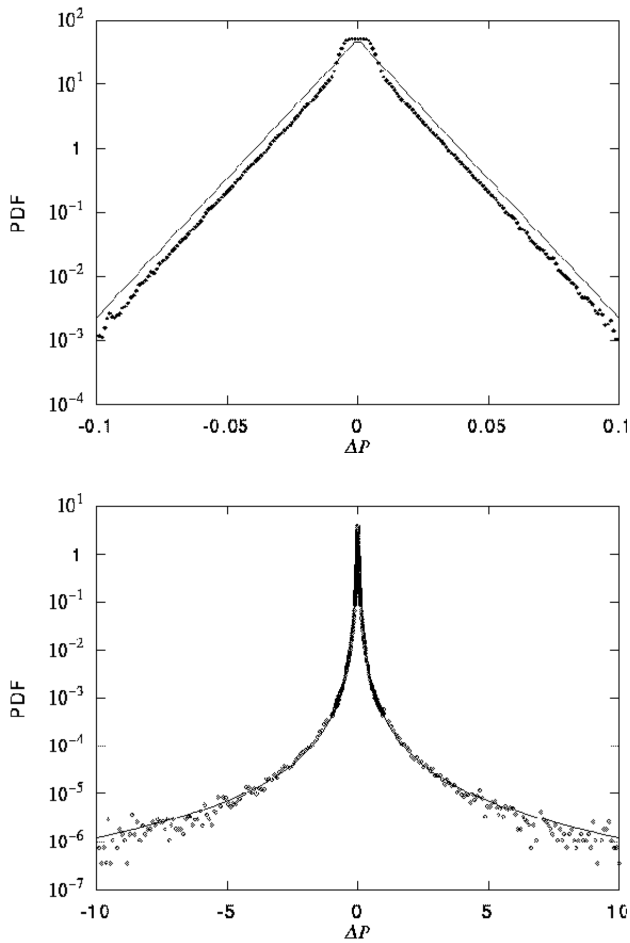

Figure 2. Semi-log plot of the PDFs of $\Delta P(t)$

the calculation of the PDF we have observed price changes for more than a million time steps. We show PDFs for $c = 0.0$ and $c = 0.3$, respectively, in Figs. 2a and 2b. For small c the PDF can be well approximated by a hybrid distribution of a Gaussian distribution for small $|\Delta P(t)|$ and of a Laplacian distribution for large $|\Delta P(t)|$. For c's larger than about 0.1 but less than 0.45, the PDF is approximated by a power law. The exponent of the power law distribution is smaller for larger c's. For c larger than 0.45, the price fluctuations are very unstable and diverge quickly, i.e. we cannot observe any steady distribution.

The PDF looks similar to the distribution of price changes for real stock markets reported by Mantegna and Stanley when c is about 0.3, except for the tail parts for very large $|\Delta P(t)|$. We will discuss the quick decay in the last section.

4. Stochastic formulation

We have seen in the preceding section that the deterministic dealer model produces seemingly stochastic fluctuations similar to the real data. By simply viewing the resulting fluctuations as stochastic fluctuations, it is shown that the time evolution can be approximated by a simple linear stochastic equation with multiplicative randomness:

$$\Delta P(t+1) = cn(t)\Delta P(t) + \phi(t), \tag{7}$$

where $n(t)$ and $\phi(t)$ are independent random numbers. Here, $n(t)$ is a natural number corresponding to the number of time steps between two successive transactions in the dealer model. The distribution function of $n(t)$ can be approximated by a discrete exponential function, meaning that the occurrence of transactions can be approximated by a Poisson process. The additive random number, $\phi(t)$, comes from the chaotic fluctuations in the dealer model with $c = 0$, namely, its PDF is given by Fig.2a.

If $n(t)$ is a constant then Eq.(7) becomes a discrete Langevin equation in which $1 - cn(t)$ is proportional to the viscosity coefficient and $\phi(t)$ corresponds to the random force.

In physical systems at equilibrium, the viscosity is always positive and the system is stable, however, in the present case of market prices, $cn(t)$ can be larger than 1 with a certain probability, which corresponds to a negative viscosity state. When the viscosity is negative, the fluctuation is enhanced, and the system becomes unstable. If the viscosity is always negative, the fluctuations diverge exponentially with time, but if the probability of taking negative values is not so big, the instability does not affect the stability of the whole system. There is a clear discussion of the criterion when the instability breaks the whole system and the fluctuation becomes nonstationary. The condition for the fluctuations to be stationary is given by [5]:

$$\log\langle cn(t)\rangle < 0. \tag{8}$$

In the case that the parameters are fitted with the preceding dealer model, the system is expected to be statistically stationary if c is less than 0.42. This estimation is consistent with the simulation results, in which the critical value is estimated to be 0.45.

Due to the linearity of Langevin type equations, the statistical properties can be solved analytically using the characteristic function method [4]. It may be proved that the statistically steady state is independent of the initial condition and the PDF of the fluctuation converges to a power law distribution. The exponent of the PDF is given by solving the equation

$$c^{\beta}\langle n(t)^{\beta}\rangle = 1, \tag{9}$$

where β satisfies

$$W(|\Delta P|) \propto |\Delta P|^{-\beta-1} \tag{10}$$

for large $|\Delta P|$. As shown in Fig.3, the theoretical curve derived by this formula fits nicely for the whole range of c's. In the mathematically rigorous sense, this formula is valid for $0 < \beta < 2$, but it is known to be a good approximation also for $\beta \geq 2$ [6]. According to Mantegna and Stanley, the distribution of averaged stock market price changes are well approximated by a symmetric power law with an exponent of about $\beta = 1.4$. The corresponding value of c can be estimated as $c = 0.28$. Within the framework of our present approach we have no explanation for this value.

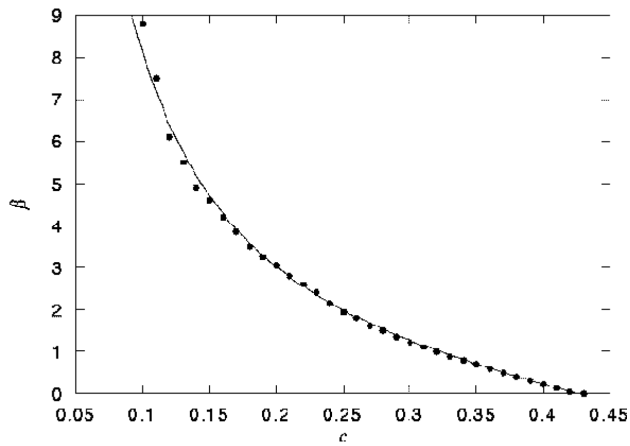

Figure 3. Relation between β and c

5. Summary and Discussion

From a physicist's point of view, the classical argument about supply and demand can be viewed as a kind of mean field theory, in which both spatial and temporal fluctuations are neglected. Taking into account the effect of stochastic fluctuations, the equilibrium point of demand and supply can be viewed as the critical point of a phase transition between two phases: the excess demand and excess supply phases. As a natural consequence of critical behaviour, we generally have a large fluctuation at the critical point. This effect is dominant for small markets with small price elasticity. With only the simplest and most natural assumptions we can easily derive a Brownian price fluctuations at equilibrium.

The basic transaction of buying and selling is characterized by its highly nonlinear and irreversible nature described by threshold dynamics. This nonlinearity can be considered as the very source of price fluctuations. Actually, as we have seen, a mathematical, deterministic model of the market consisted of dealers exhibiting chaotic fluctuations. It is confirmed that the price change in a unit time is well approximated by a Langevin type stochastic equation with random coefficients. What makes this economic model unique is that the viscocity is fluctuating near zero, and it can also take on negative values with a certain probability. When viscosity is negative in the Langevin equation, the system is unstable, and fluctuations are amplified. It is known from mathematics that if the probability of assuming negative values is finite and the steady state condition, Eq.(8), is satisfied, then there will always be power law tails in the distribution of price changes.

In the real data there is a truncation of the power law tails of the price changes. For very large values, the distribution decays more quickly than any power law. Such a cutoff effect can be easily introduced in our models. In the case of the Langevin type stochastic equation, we can introduce a kind of nonlinearity by making the multiplicative coefficient $cn(t)$ depend on $|\Delta P(t)|$. For example, we set a threshold value and modify the rule such that if $|\Delta P(t)|$ is larger than the threshold value, then $cn(t)$ cannot be larger than 1. With this modification the distribution decays quickly following a stretched exponential form for larger price changes, as expected [8].

A similar effect can also be implemented in the deterministic dealer model. A reasonable assumption is to introduce the effect of memory decay. In the case of Eq.(5), we assume that the memory of the latest price change holds until the next transaction. We modify this term by multiplying it by a factor which decays exponentially with the time interval between transactions. This modification produces a quick decay quite similar to the stochastic model [4].

We have clarified the physical mechanisms of the two most basic properties of market price changes: the spontaneous fluctuations with inverse square power spectra and the power law distributions. There is a lot of room for further study, for example, a theoretical derivation of the exponent of the power law distributions, interactions among brands, responses to external forces, predictability and controllability.

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