STOCHASTIC DYNAMICS IN GAME THEORY

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1. Introduction

The economic world is a complex many-body dynamical system the fluctuation phenomena of which have recently attracted much attention in the physics community [1, 2, 3, 4, 5]. The identification, in economics and finance, of phenomena (such as scaling and multiscaling) which also occur in physical systems (such as critical phase transitions and turbulence) suggests that some of the knowledge and techniques developed in physics to understand fluctuation phenomena, might also be useful in understanding fluctuations in economics and finance.

Fluctuation phenomena in physics depend on the nature of the equilibrium state. The starting point for understanding fluctuations in economics is then a theory for economic equilibria. Game theory [6] is the natural candidate: it describes the interaction between players' strategies and, assuming rationality, it identifies the possible game equilibria, named after Nash [7].

It is important to stress at this point that we shall deal mainly with economics and not with finance. Finance is, loosely speaking, a dynamical system out of equilibrium, where players gamble (speculate) on future market's fluctuations (see ref. [5] for a model). In an economic system, on the other hand, the assumption of rational behavior is more realistic. Still perfect rationality is utopistic in real life. Deviations from rationality, which can arise from human errors, from limited or incomplete information or from random events, are practically unavoidable. Put differently, one can say that in real life reducing human errors or the impact of random events costs time and money. Infinite precision is impossible except at infinite cost. The analysis of the effects of "irrationality" is thus an important issue in understanding how game theoretical predictions can be modified in realistic situations.

We shall address this issue for a "thermodynamic" economic system: a system with many degrees of freedom (players). The random events discussed above then have the same qualitative nature as thermal fluctuations in statistical mechanics. Indeed, one can say that in a thermodynamic system each degree of freedom pursues the minimization of the (global) energy in the presence of random shocks due to thermal fluctuations. In the same manner we shall assume that in a macro-economic system each agent pursues the maximization of his utility under the effect of random shocks. In this perspective, Nash equilibria become analogs of ground states in statistical mechanics and deviations from rationality can be introduced in exactly the same way as temperature is introduced in statistical mechanics. In particular, there are several types of dynamics depending on the nature of the problem that may be used to model temperature in statistical mechanics. This leads us to a natural definition of stochastic dynamics in game theory.

In this work, which is an extension of a previous paper [8], we shall follow these lines using the Langevin approach. We shall address the issue of fluctuations around Nash equilibria and the effects of deviations from rationality in some specific games with a large number n of players. We shall focus on games where each of the n players can control a continuous variable or "strategy" x_i. He is endowed with a utility function u_i which depends on his strategy x_i as well as that of other players $\{x_j,\ \ j \neq i\}$. In the model, each player continuously adjusts his variable x_i in order to maximize his utility. He also faces stochastic shocks which affect his actions and, as a result, the variable $x_i(t)$ becomes a continuous time stochastic process. The stochastic force acting on a player is similar to that arising from a "heat bath" at finite temperature in statistical mechanics. The dynamics allows for a comparison with equilibrium dynamics in statistical mechanics, which we find a useful paradigm for discussing the results. In this comparison negative utility plays the role of energy and the effects of fluctuations can be compared to entropic effects in statistical mechanics. The main difference between the two dynamics is that while in statistical mechanics each degree of freedom evolves to minimize a global (free) energy, in game theory each degree of freedom maximizes his own (part of the total) utility.

We shall focus on a particular class of games with a *global interaction*. By this we mean that the utility u_i of each player depends on the other players' strategies x_j only through an aggregate or global quantity $\bar{x}$, whose value is determined by all of them. This peculiar interaction, shown schematically in Fig. 1, reflects the nature of economic laws such as the law of supply and demand.

Nash equilibria are stationary points of the dynamics. When we include fluctuations we find that two main equilibria exist: *i)* non-competitive equilibria, where each player's equilibrium strategy is determined by its interac-

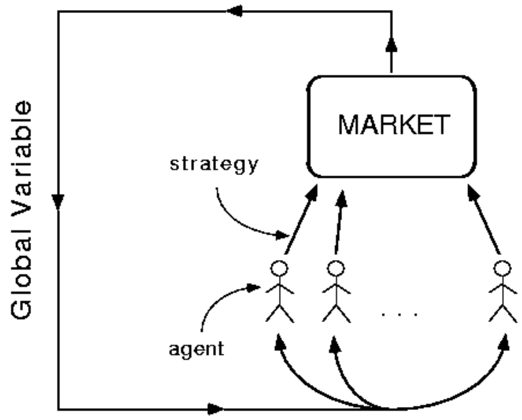

Figure 1. Global interaction among players in a stock market.

tion with the rules of the game and *ii)* competitive equilibria, which result from the the aggressive competition of each player with all the others. For example, we shall discuss a game where the introduction of taxes determines both a non-competitive and a competitive equilibrium. In the former, it will be the balance between profit and loss due to taxes, which is important for each player. In the other, competitive equilibrium, taxes are negligible and the balance leading to equilibrium is only due to the competition between all players.

The main results that we shall find are:

1. Competitive Nash equilibria are usually subject to very large fluctuations which increase with the number of players. Competition leads to broad distributions and large inequalities in an economic system. This is reminiscent of the Pareto law of distribution of incomes [9]. We shall find that inequalities increase with the number of players.
2. Competitive equilibria are also characterized by large relaxation times which are proportional to n, and by a negative correlation between players' strategies. This means that two players tend to have opposite fluctuations around the Nash equilibrium.
3. Contrary to statistical mechanics, where fluctuations always increase the system's energy, we shall find that in a game theoretical system, under particular conditions, fluctuations increase the utility (i.e. decrease the "energy").
4. We can, in principle, compute the stationary state distribution in strategy space. This allows us to solve, for example, the problem of equilibrium selection in games where more than one Nash equilibrium exists.
5. Fluctuations, in general, displace Nash equilibria and in some cases,

for strong enough randomness, a Nash equilibrium can even disappear.

6. Our approach also suggests that the time-scales for the transition from one Nash equilibrium to another are proportional to $\exp[-\Delta F/D]$, where ΔF is a "free energy barrier" and D is the noise strength.

The paper is organized as follows. The next section reviews game theory and discusses some simple games. We also discuss briefly evolutionary game theory and its differences from our approach. Sec. 4 introduces the class of models we shall analyze. The following section discusses Gaussian fluctuations around Nash equilibria. Then we develop a general approach to the stationary state probability distribution in strategy space. The main results are illustrated with simple examples. In the final section we summarize the results, we draw some conclusions and comment on possible further extensions.

2. Game Theory and Evolution

An economic system consists of a large number of interacting agents. In the game theoretical setting, each agent has a spectrum of strategies parametrized by an index x. Each player i is also endowed with a utility or payoff function u_i which depends on the strategy x_i he plays, as well as on that played by all other players. With the notation $x_{-i} = \{x_j,\ j \neq i\}$, we can conveniently write $u_i = u_i(x_i, x_{-i})$. The x_i are also called *pure* strategies as opposed to *mixed* strategies $\mu_i(x)$, in which strategy x is played with probability $\mu_i(x)$ by player i. Under mixed strategies, the strategies used by the players are *independent* random variables. Independence is justified in *one stage* games, which are played just once and each player has to decide his strategy simultaneously, without information about what the others will do.

Game theory, in its simplest setting, assumes that the payoff functions are common knowledge and that each player behaves rationally. Rationality is also common knowledge, which means that each player knows all other players are rational. Game theory aims at predicting the possible stable states of the system, which are called *Nash equilibria* [7]. The strategies x_i^* are a Nash equilibrium (NE) if each player's utility, for fixed opponents' strategies x_{-i}^*, is a maximum for $x_i = x_i^*$, i.e. $u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*)$, $\forall x_i$. The NE strategies x_i^* are such that no player has incentives to change his strategy, since any change would cause a utility loss. Nash showed [7] that any finite game has at least one Nash equilibrium in the space of mixed strategies.

2.1. THE COURNOT GAME AND THE TRAGEDY OF THE COMMONS

The concept of NE is best illustrated by a simple example, originally introduced by Cournot in 1838 [10]: 2 firms produce quantities x_1 and x_2 respectively, of a homogeneous product. The market-clearing price of the product depends, through the law of supply and demand, on the total quantity $X = x_1 + x_2$ produced: $P(X) = a - bX$. The larger X the smaller P is. The model assumes that the cost of producing a quantity x_i is cx_i and $c < a$. The firms choose their strategies (i.e. x_i) with the goal of maximizing their profit. We can then define the payoff function as $u_i = x_i[P(x_1 + x_2) - c] = x_i[a - c - b(x_1 + x_2)]$. The problem is to find x_i assuming that both firms behave rationally. One way to do this is by means of the concept of *best response*. The best response of a player i to the opponents strategies x_{-i}, is the strategy $x_i = x_i^*(x_{-i})$ which maximizes $u_i(x_i, x_{-i})$. In the Cournot game, the best response $x_1^*(x_2)$ of firm 1 to any given strategy x_2 of firm 2 is obtained by maximizing $u_1(x_1, x_2)$ with respect to x_1 with fixed x_2, i.e. $x_1^*(x_2) = (a - c - bx_2)/(2b)$. Firm 2, knowing that 1 behaves rationally (i.e. that it will play $x_1^*(x_2)$ whatever x_2 is) will choose x_2^* which maximizes $u_2(x_1^*(x_2), x_2)$. It is important to stress that operationally this means

$$\frac{\partial}{\partial x_2} u_2(x_1, x_2)\bigg|_{x_1 = x_1^*(x_2)} = 0,$$

i.e. the maximum of u_2 must be found at *fixed* x_1. This leads to $x_1^* = x_2^* = (a - c)/3b$. This simple example shows that rationality, and the assumption of others' rationality, plays a crucial role in the concept of Nash equilibrium [7]. It is easy to generalize this game to n firms. Let us set, for convenience, $a - c = 1$ and $b = 1/n$. Then $u_i(x_i, x_{-i}) = x_i(1 - \bar{x})$, where $\bar{x} = (x_1 + \ldots + x_n)/n$ is the average of x_i. The calculation generalizes straightforwardly and we find [8] a NE at $x_i = x_0 = n/(n + 1)$ and a per player payoff $u_i = n/(n+1)^2$. This celebrated example is also known as *the tragedy of the commons* [11]. In this formulation of the problem, the utility $u_i = x_i V(X)$ depends on a common resource $V(X) = c - P(X)$ which, at the NE, is almost totally depleted by the aggressive behavior of the players. As a result, each player receives a very small payoff. This is an example of a *competitive* NE where the strategies of the players are not limited because of a direct loss in utility, but because the global resource is almost exhausted by their aggressive, competitive behavior.

2.2. REPEATED GAMES AND EVOLUTIONARY GAME THEORY

This setting generalizes to repeated games, where different stage games are played a finite or an infinite number of times. In repeated games a strategy must describe the action the player has to take at each stage. Also the single stage utility is generally replaced by a utility which accounts for many stages, usually with a *discount* factor (i.e. an exponential weight function for future utility). All these things make the analysis much more complex than in single stage games.

A game posesses, in general, several NE's, and this raises the question of which of them is more relevant. In order to answer this question, several definitions of stability have been proposed [12, 13, 14]. The most successful approach to stability has been that of evolutionary game theory [15, 12]. This considers a game with mixed strategies as a game played by a population of players playing pure strategies with random opponents.

This idea has attracted much interest in the community of theoretical population biologists, who have translated it into a mathematical model, the so-called *replicator dynamics* [16]. Though several versions of this dynamics have been proposed [16, 12], in its simplest form it assumes that the individuals playing a given strategy reproduce at a rate proportional to their utility. Stochastic fluctuations in the population dynamical setting of replicator dynamics have also been considered in refs. [17].

There are some points that are worth pointing out in evolutionary game theory. The first is that its application is mostly limited to two player games. Indeed its generalization to contexts with n players is technically very complex [12]. The second is that rationality is *not* assumed. Players are, on the contrary, rather dull: they just keep playing the game the way their ancestors did. Rational NE results from the selective evolution of replicator dynamics. Finally we note that replicator dynamics assumes that the strategies x_i are *independently* chosen by each player[1]. It also assumes that $x_i(t)$ is independent of $x_i(t')$ for $t' \neq t$.

In contrast with these observations, our goal is to study a simple realistic dynamics for an $n \gg 1$ players game with "almost" rational players. We shall do this by weakening the assumption of perfect rationality with the introduction of "thermal" noise. Therefore $x_i(t)$ will become a continuous time stochastic process. We shall therefore pursue quite different goals and use totally different techniques than those of evolutionary game dynamics.

[1]The joint distribution of the strategies factorizes into the individual players' distribution functions $\mu_i(x, t)$.

3. The Langevin Approach

Focusing on a class of models with a continuum spectrum of strategies x_i, we recently proposed [8] a Langevin dynamics of the form

$$\partial_t x_i = \Gamma_i \frac{\partial u_i}{\partial x_i} + \eta_i \tag{1}$$

where the stochastic term $\eta_i(t)$ models deviation from perfect rationality. We take $\eta_i(t)$ to be a Gaussian with $\langle \eta_i(t) \rangle = 0$ and

$$\langle \eta_i(t) \eta_j(t') \rangle = 2\Delta_{i,j} \delta(t - t'). \tag{2}$$

Eq. (1) is a model dynamics which contains both the deterministic efforts each player exerts to increase his payoff and the effects of random events. The deterministic part assumes "local rationality" of the agents: each agent knows which is the direction in which his utility increases. In other words, it assumes that each agent knows the utility function u_i as a function of x_i in a small neighborhood of x_i. Note that this weakens the assumption of perfect rationality, according to which each player knows the function u_i for any x_i and any x_j, $j = 1, \ldots, n$.

The stochastic term η_i represents all hindrances which prevent rational behavior. These may be internal, i.e. affect only one player (e.g. illness), or external if they affect all players equally (e.g. earthquake). This suggests that η_i is composed of two components, $\eta_i = \tilde{\eta}_i + \tilde{\eta}_0$, where the $\tilde{\eta}_i$'s are independent Gaussian forces. This motivates our choice of

$$\Delta_{i,j} = D_i \delta_{i,j} + D_0 \tag{3}$$

for the correlation of η_i. Here D_i is the strength of the stochastic force $\tilde{\eta}_i$ acting on player i, whereas D_0 is that for events $\tilde{\eta}_0$ which affect all players in the same way.

Clearly, since NE are defined as the set of x_i for which the equations

$$\frac{\partial}{\partial x_i} u_i(x_i, x_{-i}) = 0 \tag{4}$$

are simultaneously satisfied, for $\Delta_{i,j} = 0$, NE are stationary points of the dynamics. Equation (1) is also very appealing since, comparing Eq. (1) with *model A* dynamics [18], it allows for an analogy with statistical mechanics. The main difference to statistical mechanics is that in game theory each degree of freedom (player's strategy) maximizes a different function (player's utility) whereas in statistical mechanics each degree of freedom minimizes the same function (energy or Hamiltonian). Also at odds with statistical mechanics, the interactions between strategies need not be symmetric: players' goals may be in conflict with one another.

In this analogy, NE are analogs of zero temperature (meta) stable states. The Langevin approach includes "thermal" fluctuations in game theory, and this allows the stability of NE to be analyzed through the study of fluctuations. It also gives a "free energy" measure, which enables the real "ground state" to be distinguished from "meta-stable" states. Indeed, in an ideal slow cooling down where $\Delta_{i,j} \to 0$, analogous to simulated annealing, only the state with the smallest "free energy" is selected *independently of the initial conditions*. This contrasts with the evolutionary approach, where the final state is uniquely determined by the initial conditions. The Fokker-Planck equation associated with Eq. (1) provides a description of the game at the level of the distribution of x_i. In contrast to replicator dynamics (which also involves the distribution functions of the x_i), this does not assume that the x_i are independent. As we shall see, correlations indeed arise.

4. The Model

Many complex systems in economics have a very peculiar form of interaction (see Fig. 1). In a stock market, for example, each agent guesses whether to buy or to sell a stock, looking at the stock's price fluctuations. These fluctuations are in their turn produced by the cumulative effect of the actions of all the agents in the market, through some form of supply and demand law [5]. Each player interacts with a signal, which in its turn is determined by the collective effect of all players. A further example is the above mentioned Cournot model, where n firms produce the same good, and the market clearing price is determined by the ratio between the aggregate production and the demand.

Focusing on this kind of interaction, in the following we consider situations where the payoff function for player i is

$$u_i(x_i, x_{-i}) = -B(x_i, \bar{x}), \qquad \bar{x} = \frac{x_1 + \ldots + x_n}{n}. \tag{5}$$

In other words, the payoff for player i depends on x_j for $j \neq i$ only through the *aggregate* quantity $n\bar{x}$. The n-player Cournot game is of this form with $-B(x, y) = xV(y)$, and has been discussed at length in ref. [8].

Because of the symmetry of the interaction, the NE are symmetric: $x_i^* = x^*$ for all i, and x^* satisfies the equation

$$\left.\frac{\partial u_i}{\partial x_i}\right|_{x_j = x^*} = -B_{1,0}(x^*, x^*) - \frac{1}{n} B_{0,1}(x^*, x^*) = 0 \tag{6}$$

where we defined, for convenience,

$$B_{j,k}(x,y) = \frac{\partial^j}{\partial x^j}\frac{\partial^k}{\partial y^k}B(x,y), \quad B_{0,0}(x,y) = B(x,y).$$

A NE must not only be an extremum of u_i, it must also be a maximum with respect to x_i at fixed opponents' strategies $x_{-i} = x^*$. This requires

$$\left.\frac{\partial^2 u_i}{\partial x_i^2}\right|_{x_j=x^*} = -B_{2,0}(x^*,x^*) - \frac{n+1}{n^2}B_{1,1}(x^*,x^*) - \frac{1}{n^2}B_{0,2}(x^*,x^*) < 0. \quad (7)$$

It is worth emphasizing that Eqs. (6,7) are necessary but not sufficient for x^* to be a NE. Indeed an NE must be globally stable, which means that $u_i(x_i, x_{-i} = x^*)$ must have a global minimum at $x_i = x^*$, $\forall i$.

5. Fluctuations around a Nash Equilibrium

Let $x_i = x^*$ be an NE for our model. Without loss of generality we can set $x^* = 0$ by a linear transformation $x_i \to x_i - x^*$. We shall also set $B(0,0) = 0$. We can then investigate the small Gaussian fluctuations around the NE resulting from the Langevin dynamics (1). Expanding the deterministic part to leading order, we arrive at the equation

$$\dot{x}_i = -\Gamma_i \sum_{j=1}^{n}\left(g\delta_{i,j} + \frac{h}{n}\right)x_j + \eta_i \quad (8)$$

where

$$\begin{aligned} g &= B_{2,0}(0,0) + \frac{1}{n}B_{1,1}(0,0), \\ h &= B_{1,1}(0,0) + \frac{1}{n}B_{0,2}(0,0). \end{aligned}$$

Stability requires that all the eigenvalues of the matrix $G_{i,j} = \Gamma_i(g\delta_{i,j} + h/n)$ must be positive. These are given by the equation

$$\frac{1}{h} = \frac{1}{n}\sum_{k=1}^{n}\frac{\Gamma_k}{\lambda - g\Gamma_k}. \quad (9)$$

A graphic inspection of the solutions of these equations shows that if

$$g > 0, \qquad h + g > 0, \quad (10)$$

then all eigenvalues are positive. Note that, in terms of g and h the local stability condition (7) reads $g + h/n > 0$. For $n \geq 1$ this is clearly satisfied if the conditions (10) are met.

Equation (8) is a multivariate Ornstein-Uhlenbeck process [19]. Fluctuations around the NE are described by the matrix $\sigma_{i,j} = \langle x_i x_j \rangle$ of correlations. This is the solution of the set of linear equations $G\sigma + \sigma G^T = 2\Delta$, where $G_{i,j} = \Gamma_i(g\delta_{i,j} + h/n)$ and Δ is the matrix of the noise correlation given in Eq. (3) [19]. Introducing the vector $v_i = \sum_j \sigma_{i,j}$, this matrix equation can be reduced to

$$g\sigma_{i,i} + \frac{h}{n} v_i = \frac{D_i + D_0}{\Gamma_i} \tag{11}$$

$$\left(g + h\overline{\frac{\Gamma}{\Gamma_i + \Gamma}}\right) v_i + h\overline{\frac{\Gamma_i v}{\Gamma_i + \Gamma}} = \frac{D_i}{\Gamma_i} + \overline{\frac{2nD_0}{\Gamma_i + \Gamma}}. \tag{12}$$

Here we have introduced the notation $\overline{f} = \frac{1}{n}\sum_k f_k$ for averages over the ensemble of players. Note that $v_i = n\langle x_i \bar{x} \rangle$ is the correlation between the variable x_i and the global variable $\bar{x}$.

Qualitatively there are two different cases according to whether the Γ_i are broadly distributed or not. We shall first focus on the second case, when the average value of Γ_i is much larger than the fluctuations around it: $\overline{(\Gamma - \overline{\Gamma})^2} \ll \overline{\Gamma}^2$. With a redefinition of the scale of B, we set, for simplicity, $\overline{\Gamma} = 1$. In the limit $n \to \infty$ and $\epsilon = \sqrt{\overline{(\Gamma - \overline{\Gamma})^2}} \ll 1$, the values of Γ_i are densely distributed in a small interval of size $\approx \epsilon$. In each interval $[\Gamma, \Gamma + d\Gamma)$, $d\Gamma \ll \epsilon$ we can define an average value of D_i, $\sigma_{i,i}$ and v_i, which we denote by $D(\Gamma)$, $\sigma(\Gamma)$ and $v(\Gamma)$. This allows for a systematic expansion in powers of ϵ.

For $\epsilon = 0$, one easily finds

$$\begin{aligned} v(1) = \overline{v} &= \frac{\overline{D} + nD_0}{g + h}, \\ \sigma(1) = \overline{\sigma} &= \left(1 - \frac{h}{n(g+h)}\right)\frac{\overline{D}}{g} + \frac{D_0}{g+h} \end{aligned} \tag{13}$$

Note that $v(1) > 0$, which means that each variable x_i tends to fluctuate in phase with $\bar{x}$. We can also compute an ensemble average of the correlation, which for $\epsilon = 0$ reads

$$C = \frac{1}{n(n-1)} \sum_{i \neq j} \langle x_i x_j \rangle = \frac{\overline{v - \sigma}}{n-1} = \frac{D_0}{g+h} - \frac{h\overline{D}}{ng(g+h)}. \tag{14}$$

The common stochastic force, as could be expected, gives a positive contribution to the correlation, and for D_0 large enough, the correlation always turns positive.

With respect to the dynamics in the stationary state, correlation functions decay exponentially

$$\langle x_i(t+t_0)x_j(t_0)\rangle - \sigma_{i,j} \propto e^{-t/\tau} .$$

The correlation time τ of the leading exponential behavior is given by the minimum eigenvalue $\tau = \max_k(\lambda_k^{-1})$ in Eq. (9).

It is finally possible to compute the average utility

$$\overline{\langle u\rangle} = -\frac{1}{2}\overline{\sigma}B_{2,0} - \frac{\overline{v}}{n}B_{1,1} - \frac{\overline{v}}{2n}B_{0,2} \tag{15}$$

where all derivatives of B are evaluated at $(0,0)$. Note that in view of our choice $B(0,0)=0$ this expression yields the deviation of the average utility from completely rational behavior. As we shall see, it is possible that fluctuations increase the average utility. The last term in Eq. (15) is the average of the utility *at* the Nash equilibrium *in the presence* of fluctuations:

$$\langle u_i(0,x_{-i})\rangle = -\frac{\overline{v}}{2n}B_{0,2}.$$

This would be the utility of a player which maintains the NE strategy $x_i=0$ even in the presence of fluctuations. It is interesting to note that it is possible that $\overline{\langle u\rangle} > \langle u_i(0,x_{-i})\rangle$. Loosely speaking this means that in a game with random deviations from rationality players who are affected by the randomness may receive a higher payoff (on average) than those who behave rationally ($x_i=0$). The condition for this to happen is

$$\frac{1}{2}\overline{\sigma}B_{2,0} + \frac{\overline{v}}{n}B_{1,1} < 0\,. \tag{16}$$

Let us now discuss these findings. As expected, the fluctuations of x_i grow with D_i and D_0. There are three qualitatively different cases, as shown in Fig. 2:

a) when $B_{2,0}, B_{1,1}$ and $B_{0,2}$ are all finite and positive. The point (g,h) lies well inside the domain defined by Eq. (10). As a result we have normal behavior with small fluctuations. Fluctuations decrease the average utility and rational behavior is generally more rewarding.

b) $B_{2,0}\simeq 0$, $B_{1,1}$ and $B_{0,2}$ are finite and positive. Then $g\sim 1/n$ is small and, from Eq. (13), we see that fluctuations are proportional to n. This, in view of the explicit factor g in front of $\sigma_{i,i}$ in Eq. (11), is a general feature which also holds for broadly distributed Γ_i. The condition $B_{2,0}\simeq 0$ obtains for example for utilities of the form $B(x,y)=-xb(y)$, which describes e.g. the tragedy of the commons problem [11, 8]. We shall discuss in more detail this class of models in the next paragraph.

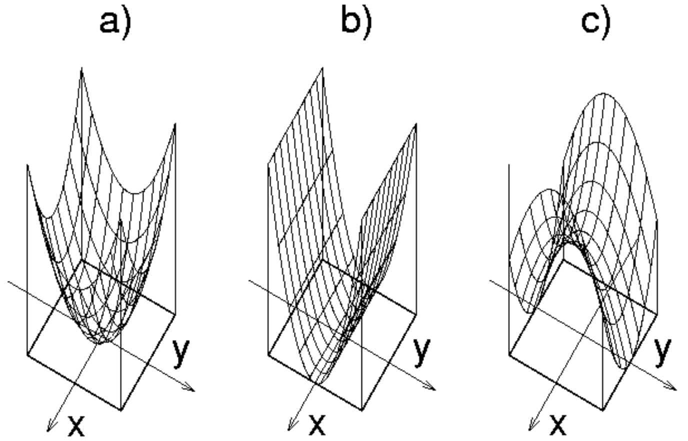

Figure 2. Quadratic approximation of the utility function close to a Nash equilibrium. a) $B_{2,0}, B_{1,1}$ and $B_{2,0}$ are finite and positive; b) $B_{2,0} \simeq 0$, $B_{1,1} > 0$ and $B_{2,0} > 0$; c) $B_{2,0} + B_{1,1} \simeq 0$.

Generally $B_{2,0} = 0$ is typical of competitive equilibria. Indeed, it means that players do not feel the effects of a change in their x_i's directly. Instead, they are affected by it indirectly, through the reaction of other players, or rather through the effects this change has on the global variable $\bar{x}$. Large fluctuations are a result of the fact that the dependence of $\bar{x}$ on x_i is weak. Competitive equilibria are also characterized by negatively correlated variables x_i for D_0 small enough: $C < 0$. Large fluctuations correspond to large relaxation times. Indeed eigenvalues are proportional to g, so that for $g \ll 1$ all of them are small, yielding large relaxation times $\tau \sim n$. Finally the average utility is decreased.

c) $B_{2,0}+B_{1,1} \simeq 0$. In this case too, large fluctuations occur since $g+h \simeq 0$. The divergence of the terms proportional to D_0 suggests that the mode with stronger fluctuations is associated with $\bar{x}$. For $h+g$ small the smallest eigenvalue is small. This results in a large correlation time $\tau = \overline{\Gamma^{-1}}/(g+h) + O(1)$. At odd with case *b)*, only one eigenvalue is small in this case, the others being $O(1)$. The (right) eigenvector associated with this eigenvalue is $w_i = 1 + O(g+h)$ nearly constant. The slow mode characterized by large fluctuations is thus associated with $\bar{x}$. Also note that $C \sim (g+h)^{-1}$ is large and positive, which means that the x_i fluctuate in phase, thus yielding a large fluctuation of their sum. Fluctuations may *decrease* the average utility in this case. Furthermore if player i behaves rationally ($x_i = 0$) he receives a smaller payoff $\overline{\langle u \rangle} > \langle u_i(0, x_{-i}) \rangle$.

These results can be extended to higher order in ϵ. The idea is to assume that Γ_i are distributed around 1 according to a Gaussian density with standard deviation ϵ. We shall limit our discussion to the first term here. Taking the average of Eq. (12), multiplied by $\Gamma_i - 1$ over this distribution and taking the leading order in ϵ, gives

$$\begin{aligned} v'(1) &= -\frac{D(1)}{g+h} + \frac{2D'(1)}{2g+h} - \frac{gnD_0}{(g+h)(2g+h)} \\ \sigma'(1) &= \frac{D'(1) - D(1) - D_0}{g} - \frac{h}{gn} v'(1) \,. \end{aligned}$$

These equations provide interesting information. For example $\sigma'(1) < 0$ implies that the fluctuations experienced by a player are smaller the faster his dynamics (i.e. the larger his rate constant Γ). This is what one naturally expects and it occurs when $D'(1) < D(1) + D_0[1 - gh/(g+h)(2g+h)] + O(1/n)$. On the other hand, if $D(\Gamma)$ grows sufficiently fast with Γ, one has $\sigma'(1) > 0$. This suggests that generally the fluctuations of a variable x_i grow with D_i and decrease with Γ_i. The same kind of information can be obtained for the correlation C. The case $D_0 = 0$ yields a compact expression:

$$\frac{C'}{C} = -1 + \frac{g+h}{2g+h}\frac{D'}{D}.$$

This says that correlations are weaker for faster variables, unless $D(\Gamma)$ grows fast enough with Γ. We shall not discuss the case $D_0 > 0$, which leads to less transparent formulae.

The case of broadly distributed Γ_i needs a more detailed knowledge of the parameters. However we expect that the results obtained by the small disorder expansion above qualitatively describe the system. Note that Eq. (11) suggests that $\langle x_i^2 \rangle \propto D_i/\Gamma_i$ diverges as $\Gamma_i \to 0$. In a large system of players with broadly distributed Γ_i, the smallest Γ_i can be vanishingly small as $n \to \infty$. For example, in a system where the Γ_i are distributed with a density $\rho(\Gamma) \sim \Gamma^{\beta-1}$ for $\Gamma \ll 1$, one expects that the smallest Γ_i, in the population of players, is $\Gamma_{\min} \sim n^{-1/\beta}$. In this case some player can have fluctuations growing with n. It is worth stressing, however, that such a distribution of Γ_i implies a power law distribution of characteristic times $1/\Gamma_i$, which may not be realistic.

5.1. AN EXAMPLE

Let us illustrate these findings with simple examples. An example with a utility of the form $B(x,y) = xy$ has been discussed in detail elsewhere [8]. The main point raised by this example is that of the emergence of large fluctuations. Here we shall describe an other example:

$$B(x,y|\rho) = \frac{b}{2}(x - \rho y)^2.$$

The utility function $u_i(x_i, x_{-i}) = -B(x_i, \bar{x}|\rho)$ above describes a classical game where each player has to throw a number x_i with the aim of guessing a fraction ρ of the average $\bar{x}$ of all players' guesses. This clearly has only one NE $x_i = 0$, $\forall i$. $B(x, y|\rho)$ can also be regarded as a local approximation of a more complex utility around one of its Nash equilibria.

With the choice $b = (1-\rho/n)^{-1}$, the parameters are $g = 1$ and $h = -\rho$. The stability condition requires that $\rho < 1$, which is intuitively right, because if players were told to guess a number larger than the mean, everybody would tend to overshoot and $x_i \to \infty$. On the contrary with $\rho < 1$ every player has to be careful: he must play a number which is smaller than the one played by the others. In the extreme case $\rho < 0$ he has to try to do the opposite of what the majority does. Let us discuss only the results for $\epsilon = 0$. It is straightforward to find

$$\begin{aligned} \sigma &= \overline{D}\left(1 + \frac{\rho}{n(1-\rho)}\right) + \frac{D_0}{1-\rho} \\ C &= \frac{\rho\overline{D}}{n(1-\rho)} + \frac{D_0}{1-\rho} \end{aligned}$$

Note that, for $D_0 = 0$, C has the same sign as ρ. For $\rho > 0$, a player attempts to guess the fluctuation of others and as a result he tends to make fluctuations in the same direction as the others. On the other hand, for $\rho < 0$ a negative correlation arises.

The average utility is $\overline{\langle u \rangle} = -\frac{1}{2}\overline{D} - \frac{n}{2}\frac{1-\rho}{n-\rho}D_0$ whereas the condition (16), after some algebra, reads:

$$[n - (n+1)\rho]\overline{D} + (1-2\rho)D_0 < 0\,.$$

In order for this condition to hold at least one of the two terms must be negative. For $\frac{1}{2} < \rho < \frac{n}{n+1}$, it becomes "favorable" to follow the random force for

$$D_0 \geq \frac{n - (n+1)\rho}{n(2\rho - 1)}\overline{D}\,.$$

In other words, if the global component of the stochastic force is strong enough, it is not convenient to play the NE strategy $x_i = 0$.

This behavior can be qualitatively explained as follows: Each player has to try to follow as closely as possible the global variable $\rho\bar{x}$. The latter evolves under a stochastic force $\bar{\eta}$ of strength $\overline{D}/n + D_0 \simeq D_0$. The random force η_i acting on each player has a component of strength D_0 along $\rho\bar{x}$. If this component is large enough, each player can guess correctly whether the mean $\bar{x}$ will move left or right and so it becomes favorable to follow the stochastic force.

For $\rho > \frac{n}{n+1}$ the condition holds $\forall \overline{D} > 0$ with $D_0 = 0$. This region is also characterized by large correlated fluctuations (note that $g+h = 1-\rho \sim 1/n$). Even in the absence of a global stochastic force, the dynamics leads to a state where the x_i are highly correlated. In such a state, the strategy $x_i = 0$ is less rewarding than the average.

For $\rho < \frac{1}{2}$ there are no values of $\overline{D}$ and D_0 for which the condition (16) is satisfied.

6. Probability Distribution in Strategy Space

In this section we shall extend the analysis of our model to study the full probability distribution in the stationary state. A complete treatment in general is not possible. We shall restrict attention to the case

$$\Gamma_i = 1, \qquad D_i = D\,.$$

In view of our discussion of the previous section, $\Gamma_i = 1$ means that all players have the same characteristic time-scale. Qualitatively, we expect that the results below apply also in the case of narrowly distributed time-scales.

Our equation is

$$\dot{x}_i = -B_{1,0}(x_i, \bar{x}) - \frac{1}{n} B_{0,1}(x_i, \bar{x}) + \eta_i. \tag{17}$$

It is useful to introduce the variables

$$y_k = \frac{1}{\sqrt{k(k+1)}} \sum_{i=1}^{k} (x_i - x_{k+1}), \qquad k < n \tag{18}$$

$$y_n = \frac{x_1 + \ldots + x_n}{\sqrt{n}} = \sqrt{n}\bar{x}. \tag{19}$$

The transformation $\vec{x} \to \vec{y}$ is orthonormal, which implies the useful identity

$$\sum_{i=1}^{n} x_i^2 = \sum_{k=1}^{n} y_k^2. \tag{20}$$

The same transformation applied to the noise term $\vec{\eta} \to \vec{\zeta}$ leads to a stochastic force ζ_k in the equation for $\dot{y}_k$ which has a correlation

$$\langle \zeta_j(t) \zeta_k(t') \rangle = 2\delta_{j,k}(D + nD_0\delta_{k,n}) \tag{21}$$

which is diagonal. The common stochastic force D_0 acts on y_n only. For convenience, instead of y_n, we shall use the variable $\bar{x} = y_n/\sqrt{n}$. The noise $\bar{\eta}$ appearing in the equation for $\dot{\bar{x}}$ has a strength $T = D/n + D_0$.

6.1. LINEAR UTILITY

Let us first consider the model

$$B(x, z) = xb(z),$$

which allows for a full solution for the stationary state distribution of x_i. A situation described by this kind of utility function [11, 8] is a system where n firms produce a quantity x_i of a homogeneous product. Then $-b$ is the difference between the market clearing price of one unit of product and the production cost of one unit. We assume that it depends only on the aggregate production $\sum_i x_i = n\bar{x}$ (the production cost per unit is a constant). The payoff $u_i = -x_i b$ of firm i is then proportional to its production. In realistic situations $b(x)$ is an increasing function. One expects e.g. that because of competition, the price $-b$ of a product decreases with the total quantity produced, in view of the law of supply and demand.

The NE $x_i = x^*$ is defined by $b(x^*) = -x^* b'(x^*)/n$. Note that the payoff per player

$$u_i = -x^* b(x^*) = \frac{x^{*2} b'(x^*)}{n}$$

is positive and proportional to $1/n$.

The orthonormal transformation $\vec{x} \to \vec{y}$, yields

$$\dot{y}_k = -\frac{b'(\bar{x})}{n} y_k + \zeta_k, \qquad k < n \tag{22}$$

$$\dot{\bar{x}} = -b(\bar{x}) - \frac{b'(\bar{x})}{n} \bar{x} + \bar{\eta}. \tag{23}$$

The equation for $\bar{x}$ does not involve other variables and can be directly solved yielding the distribution $p_n(\bar{x})$. The equations for y_k depend only on $\bar{x}$. Treating $\bar{x}$ as a parameter, one can find the conditional distribution of y_k: $p(y_k|\bar{x})$. The full distribution is then

$$p(y_1, \ldots, y_n) = p_n(\bar{x}) \prod_{k=1}^{n-1} p(y_k|\bar{x}).$$

Transforming back to the variables x_i yields the solution. Eq. (23) describes a "particle" in a potential with thermal fluctuations and can be solved using standard techniques [19]:

$$p_n(\bar{x}) = \mathcal{N} \exp\left\{ -\frac{\bar{x} b(\bar{x}) + (n-1) \int_0^{\bar{x}} dz b(z)}{D + nD_0} \right\} \tag{24}$$

with $\mathcal{N}$ a normalization factor. The equation for y_k similarly gives

$$p(y_k|\bar{x}) = \sqrt{\frac{b'(\bar{x})}{\pi n}} \exp\left\{-\frac{b'(\bar{x})y_k^2}{nD}\right\} \tag{25}$$

where normalization requires $b'(\bar{x}) > 0$. Using equation (20) one can easily find the full distribution of x_i:

$$p(x_1,\ldots,x_n) \propto \left[\frac{b'(\bar{x})}{\pi nD}\right]^{\frac{n-1}{2}} \exp\left\{-\frac{b'(\bar{x})}{nD}\sum_{i=1}^{n}(x_i-\bar{x})^2 \right.$$
$$\left. - \frac{\bar{x}b(\bar{x}) + (n-1)\int_0^{\bar{x}} dz b(z)}{D+nD_0}\right\}$$

where $\bar{x} = (x_1 + \cdots + x_n)/n$.

Some implications of this result have already been discussed in ref. [8]. In particular it was observed that if $b'(x) \sim O(1)$ one finds fluctuations of order $\langle x_i^2\rangle \propto nD$ and large relaxation times $\tau \sim n$. We note furthermore that $p(x_1,\ldots,x_n)$ vanishes as $b(\bar{x}) \to 0^+$. For $b'(\bar{x}) < 0$, which corresponds in any case to an "unphysical" situation[2], we must set $p(x_1,\ldots,x_n) = 0$. Note that Eqs. (22, 23) imply that if the system is "prepared" at $t = 0$ in a state with $b(\bar{x}) < 0$, in the early stages of the dynamics the deviations $x_i - \bar{x}$ increase exponentially. This is clearly a highly non-equilibrium situation.

The average utility, to order $D + nD_0$, is given by

$$\langle u_i\rangle \simeq -x^*b(x^*) - \frac{[2b'(x^*) + x^*b''(x^*)](D+nD_0)}{2[(n+1)b'(x^*) + x^*b''(x^*)]}\,. \tag{26}$$

If $\frac{\partial^2}{\partial x^2}[xb(x)]_{x=x^*} < 0$, then the effect of fluctuations will be that of increasing the average utility (note that, in the notations of the previous section, $(n+1)b'(x^*)+x^*b''(x^*) = n(g+h) > 0$). An example, which allows for simple expressions, is $b(x) = -1+x-\frac{1}{2}ax^2$. Since $b'(x) = 1-ax$ we need to restrict the range of x to $x < 1/a$ in order for $b'(x) > 0$ (otherwise $\bar{x}$ would flow to ∞). The NE is at $x^* = \frac{n+1}{a(n+2)}\left(1-\sqrt{1-2a+\frac{2a}{(n+1)^2}}\right) \simeq \frac{1}{a}(1-\sqrt{1-2a})$ and its existence requires $a \le 1/2$. The payoff per player at the NE, to leading order in n, is $-x^*b(x^*) \simeq \frac{1}{na^2}(1-\sqrt{1-2a})^2\sqrt{1-2a}$. With gaussian fluctuations, we find:

$$\langle u_i\rangle \simeq -x^*b(x^*)\left[1-\frac{a^2(3\sqrt{1-2a}-1)(D+nD_0)}{2(1-2a)(1-\sqrt{1-2a})^2}\right].$$

[2] For example, in the firms' problem, $b'(\bar{x}) < 0$ means that the price increases if production is increased.

For $a \geq 4/9$ fluctuations increase the average utility, an effect related to the asymmetry of the function $b(x)$ around x^*.

The average $\langle \bar{x} \rangle$, however, has no corrections to order D. As we shall see this does not hold in general. Neither does it hold when the function $b(x)$ changes rapidly close to the NE, a situation which cannot be described in the Gaussian approximation. Consider for example, the game of the tragedy of the commons, where the utility function is much steeper when negative payoffs arise: $b(x) = x - 1$ for $x < 1$ and $b(x) = q(x-1)$ for $x > 1$ and $q \gg 1$. This represents a situation where each player is very scared of receiving negative payoffs. Clearly, as far as the NE is concerned, no change occurs: The NE $x^* = n/(n+1)$ is always the same, dangerously close to the edge of negative utilities. In the presence of fluctuations, however, the distribution of $\bar{x}$ is very asymmetric on the two sides of the NE. For $\bar{x} > x^*$ it drops off much more quickly than for $\bar{x} < x^*$. As a result the NE is shifted by an amount of order $\sqrt{D/n}$ towards safer values of $\bar{x} < x^*$. We see then that fluctuations can induce more cautious behavior.

The most general model which allows for a complete solution with the above technique is $B(x,z) = B_0(z) + xb(z) + cx^2$. The term $B_0(z)$ changes only the distribution of $\bar{x}$ by a factor proportional to $\exp[-B_0(\bar{x})/n]$, whereas the term cx^2 also affects the distribution of x_i. A simple realization of this model, with $c = 0$, is the one where players cooperate to increase each other's utility: $B_0(z) = \beta z b(z)$ $(\beta > 0)$. Of course $\beta < 0$ means anti-cooperation, i.e. each player tries to maximize his utility as well as to minimize that of others. With $b(z) = z - 1$, one easily finds that the NE is at $x^* = \frac{n+\beta}{n+2\beta+1}$ and the payoff per player is $u_i^* = \frac{(n+\beta)(1+\beta)}{(n+2\beta+1)^2}$. A high degree of cooperation, $\beta \propto n$ leads to a finite utility per player. On the other hand, fluctuations always remain large $\langle \delta x_i^2 \rangle \sim nD$. For $\beta \to \infty$, as discussed in [8], fluctuations diverge even though the average utility remains finite. Clearly anti-cooperation $\beta < 0$ decreases the utility. However for $\beta < -1$ fluctuations give a positive contribution to the utility. Fluctuations increase the average utility in over-competitive systems (those in which each player main efforts are devoted to decreasing other players' payoffs).

6.2. THE GENERAL CASE

The general case does not allow for a full solution. It is however possible to compute the stationary state distribution to leading order in D. We assume that

$$y_k \sim x_i - \bar{x} \sim O(\sqrt{D}). \tag{27}$$

While this is surely satisfied close to an NE (when all x_i are close to x^*), it might not hold in non-equilibrium situations or when rare events such as large fluctuations occur.

The equation for $\dot{y}_k$, derived from Eq. (17), contains terms $B_{j,k}(x_i, \bar{x}) - B_{j,k}(x_{k+1}, \bar{x})$ with $(j,k) = (1,0)$ or $(0,1)$. Expanding in powers of $x_i - \bar{x}$ and $x_{k+1} - \bar{x}$, we find, to leading order

$$\dot{y}_k = -\left[B_{2,0}(\bar{x},\bar{x}) + \frac{1}{n}B_{1,1}(\bar{x},\bar{x})\right] y_k + \zeta_k$$

where ζ_k is still a Gaussian uncorrelated noise, in view of the orthonormality of the transformation $\vec{x} \to \vec{y}$ (see Eq. (21)). This equation is valid to $O(D)$, since we neglected terms proportional to $(x_i - \bar{x})^2$ which are of order D in view of Eq. (27). Using Eq. (21), one easily finds:

$$\langle y_k^2 | \bar{x} \rangle = \frac{nD}{nB_{2,0}(\bar{x},\bar{x}) + B_{1,1}(\bar{x},\bar{x})} \tag{28}$$

where we used the notation $\langle y|x \rangle$ for the average of the quantity y conditional to the value x assumed by a second variable. Note that the requirement $\langle y_k^2|\bar{x}\rangle \geq 0$, implies $nB_{2,0}(\bar{x},\bar{x}) + B_{1,1}(\bar{x},\bar{x}) \geq 0$. This condition, which generalizes the condition $b'(\bar{x}) \geq 0$ in the previous paragraph, restricts the range of possible values of $\bar{x}$.

Let us now move to the equation for $\bar{x}$. Expanding the right hand side of the equation for x_i to second order in $x_i - \bar{x}$, we find

$$\begin{aligned}\dot{\bar{x}} = & - B_{1,0}(\bar{x},\bar{x}) - \frac{1}{n}B_{0,1}(\bar{x},\bar{x}) \\ & - \frac{1}{2}\left[B_{3,0}(\bar{x},\bar{x}) + \frac{1}{n}B_{2,1}(\bar{x},\bar{x})\right] \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 + \ldots + \bar{\eta}\end{aligned}$$

where $\langle \bar{\eta}(t)\bar{\eta}(t')\rangle = 2(D/n + D_0)\delta(t - t')$. In view of Eq. (20), we have $\sum_{i=1}^n (x_i - \bar{x})^2 = \sum_{k=1}^{n-1} y_k^2$. Taking the average over y_k conditional to the value of $\bar{x}$ in this equation, we substitute $\frac{1}{n}\sum_i (x_i - \bar{x})^2$ with $\frac{n-1}{n}\langle y_k^2|\bar{x}\rangle$. This results in the equation

$$\dot{\bar{x}} = -B_{1,0} - \frac{1}{n}B_{0,1} - \frac{(n-1)D}{2n}\frac{nB_{3,0} + B_{2,1}}{nB_{2,0} + B_{1,1}} + \bar{\eta}$$

where we suppressed the dependence on $(\bar{x},\bar{x})$ of $B_{k,l}$.

The steady state distribution of $\bar{x}$ is then given by $P(\bar{x}) \propto \exp[-F(\bar{x})/T]$, where the "temperature" is $T = D/n + D_0$, and the "free energy" is given by:

$$F(x) = \int^x \left[B_{1,0} + \frac{1}{n}B_{0,1} + \frac{(n-1)D}{4n}\frac{nB_{3,0} + B_{2,1}}{nB_{2,0} + B_{1,1}}\right] \tag{29}$$

It is useful for the following discussion to split $F = U - DS$ into a D independent part and into a D dependent one:

$$U(x) = \int^x \left[B_{1,0}(y,y) + \frac{1}{n}B_{0,1}(y,y)\right] dy$$

and

$$S(x) = -\frac{n-1}{2n}\int^x dy \frac{nB_{3,0}(y,y) + B_{2,1}(y,y)}{nB_{2,0}(y,y) + B_{1,1}(y,y)}$$

The relation $F = U - DS$ is reminiscent of the free energy in equilibrium statistical mechanics, which is a useful paradigm for discussing our system.

6.3. DISCUSSION AND APPLICATIONS

It is worth stressing that the function $U(x)$ is not simply related to the utility. This contrasts with equilibrium statistical mechanics, where the probability of a state is directly related to its energy.

The "entropic" term $S(x)$ results from the inclusion of the fluctuations of the variables y_k. Usually in statistical mechanics one finds that the larger the fluctuations in a state, the greater the entropy. We shall see in the following that this does not hold for our system: S can be larger for "ordered" states than for states with wild fluctuations.

Fluctuations displace the NE (defined as the minima of $F(x)$) of a quantity of order D:

$$x^*(D) = x^*(0) - \frac{(n-1)D(nB_{3,0} + B_{2,1})}{2n^2(nB_{2,0} + B_{1,1})[nB_{2,0} + (n+1)B_{1,1} + B_{2,0}]}.$$

Note that stability requires that the denominator be positive.

The analysis of any particular case goes as follows. First one needs to determine the range of $\bar{x}$ where our approach can be applied. This is given by the condition $nB_{2,0}(\bar{x},\bar{x}) + B_{1,1}(\bar{x},\bar{x}) > 0$ which ensures that $\langle y_k^2|\bar{x}\rangle > 0$ is finite. Outside this range, the behavior cannot be described perturbatively in D (i.e. the Gaussian approximation for the variables y_k is no longer valid). Secondly, find all solutions of Eq. (6), $nB_{1,0} + B_{0,1} = 0$, which are the candidates for NE's. Each solution to this equation must then be checked for stability. If Eqs. (10) are satisfied, the equilibrium is stable. Finally for each stable equilibrium one can evaluate its free energy $F(x^*)$ from Eq. (29). This gives the statistical weight of each state in the stationary regime. The state with the smallest $F(x^*)$ is the one with the largest statistical weight and it dominates in the limit $T = D/n + D_0 \to 0$.

In ref. [8] we discussed the case

$$B(x,y) = -x(1-y)\left[1 - \frac{x(1-y)}{2c}\right], \qquad c > 0 \tag{30}$$

of a quadratic utility both in x and y. This is an interesting case, both because such a utility function can be motivated [8] and because the system possesses two NE's. As a rough motivation, we can go back to the firms' problem with $b(y) = -1+y$, and argue that their utility u_i may not exactly equal their net gain $x_i(1-\bar{x})$ since this is then subject to taxes. If taxes do not grow linearly with income (as is usually the case) we need to add a further term to the utility. The simplest choice leads to the above form of $B(x, y)$.

Let us go through the above steps for this example: Eq. (6) has only one solution for $c > 1/4$ which is at

$$x_0 = \frac{n}{n+1}\,, \tag{31}$$

which is stable $\forall c > 0$. For $c < 1/4$ two other solutions appear at

$$x_\pm = \frac{1 \pm \sqrt{1-4c}}{2} \tag{32}$$

of which only x_- is stable. These two NE's have quite different natures. The NE at x_0 is a competitive NE since $B_{2,0} \ll 1$ and it is characterized by a small payoff per player $u_i \sim 1/n$ and by large fluctuations $\langle \delta x_i^2 \rangle \sim n/D$. The NE at x_- is *Pareto dominant*[3] since it has a finite and positive utility. Also fluctuations are finite, as $n \to \infty$, at x_-. At this NE the action of players is limited by the increase of the non-linear term due to "taxes".

The function $F(x)$ is readily computed. Setting, for convenience, $F(x_+) = 0$ we find

$$F(x_-) = -\frac{(1-4c)^{3/2}}{6c} + \frac{(1-4c)^{3/2}}{6c}\frac{1}{n} - \log\left[\frac{1-2c-\sqrt{1-4c}}{1-2c+\sqrt{1-4c}}\right]\frac{D}{n} + O(n^{-2})$$

and

$$\begin{aligned} F(x_0) = {} & \frac{1-6c+6c^2-(1-4c)^{3/2}}{12c} - \frac{1-6c-6c^2-(1-4c)^{3/2}}{12c}\frac{1}{n} + \\ & - \log\left[n\frac{1-2c-\sqrt{1-4c}}{2c}\right]\frac{D}{n} + O(n^{-2}). \end{aligned}$$

The function $F(x)$ is plotted in Fig. 3 for $c = 0.1$ as a function of D. The "entropic" contribution, $S(x)$ is of order $1/n$. This is a consequence of the fact that $B_{3,0} = 0$ in this model. Since $B_{2,1}(y, y) = -2(1-y)/c < 0$, fluctuations in y_k increase the probability of the state x_0. Indeed for

[3]The NE with the largest utility is called the *Pareto dominant* equilibrium in game theory.

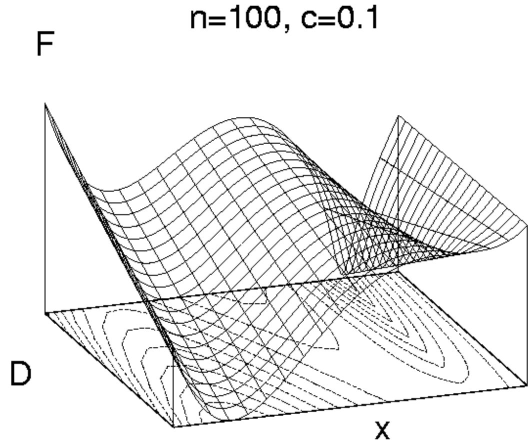

Figure 3. $F(x)$, as a function of D for the quadratic utility function: $n = 100$ and $c = 0.1$.

large enough D, Fig. 3 shows that the state at x_0 has a higher probability. Therefore fluctuations in this case *decrease* the probability of the Pareto dominant NE. Note also that as $D \to 0$ and $n \to \infty$, $F(x_0) < F(x_-)$ for $c > 2/9 = 0.2222\ldots$, which implies that the probability of finding the system in the Pareto dominant NE x_- tends to 0. This example shows that the system does not always choose the state of maximum utility. In addition, when $D > 0$ one stable state can become unstable. In our example, for higher values of D or c the state at x_- which is a minimum of $U(x)$ is no longer a minimum of $F(x)$.

In this system, however, entropic effects are "accidentally" small because $B_{3,0} = 0$. If one adds a higher order term the situation changes. Consider indeed

$$B(x,y) = -x(1-y)\left[1 - \frac{1-bc^3}{2c}x(1-y) - \frac{b}{4}x^3(1-y)^3\right], \qquad c > 0\,. \quad (33)$$

For $0 < b < \frac{4}{c^3}$ this has the same stable equilibria as before[4]. The picture in the limit $n \to \infty$ is qualitatively the same apart from the entropy, which is now finite since $B_{3,0}(x,x) = 6bx(1-x)^4 \neq 0$. Note that $B_{2,1} < 0$, and $B_{3,0} > 0$. Therefore the effects of fluctuations are opposite to the ones discussed above. Indeed $S(x_0) < S(x_-)$, which means that the stochastic force favours in this case the Pareto dominant NE x_- over the state x_0. Contrary to our intuition from statistical mechanics, it is the "ordered"

[4]For $b < 0$ the system is unstable and for $b > 4c^{-3}$, x_- becomes unstable and two new equilibria appear.

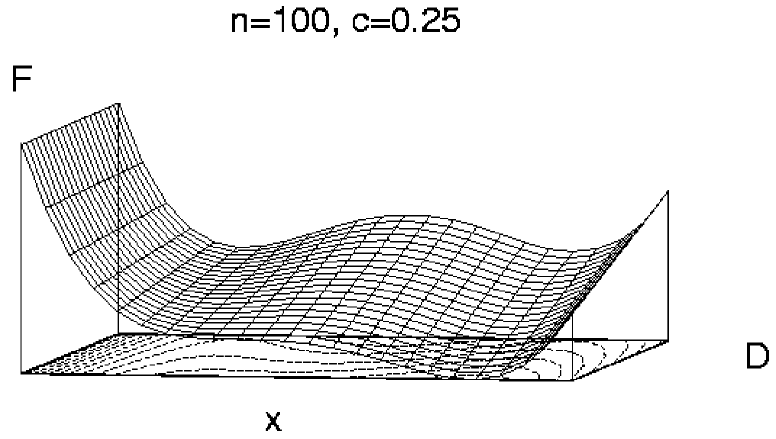

Figure 4. $F(x)$, as a function of D for the utility function (33): $n = 100$, $c = 0.25$ and $b = 0.1$.

state x_- which has a larger contribution from the stochastic term. This is shown in Fig. 4, which shows in particular that fluctuations lead to a new minimum of $F(x)$: This meta-stable state is a precursor of the NE at x_-. This example shows that the identification of $S(x)$ with an entropy can be misleading.

Direct numerical simulations of the Langevin equation gives results in good agreement with this picture for small values of D. The higher order terms in the D expansion seem to enhance the behavior discussed above for the two particular models.

For a particle in a random potential $F(\bar{x})$ subject to a stochastic force of strength $T = D/n + D_0$, the transition times from one metastable state to the other are of the order of $\tau \approx \exp\{n[F(x_i) - F(x_+)]/(D + nD_0)\}$ where $i = 0, -$ labels the state of departure. The generalization of this result to our case predicts relaxation times that, for $D_0 = 0$, diverge in the "thermodynamic" limit $n \to \infty$. This behavior is reminiscent of a first order phase transition in statistical mechanics. It is worth recalling, however, that the transition from one state to the other is a far from equilibrium process, whereas we derived Eq. (29) within an approximation which is valid to order D close to the equilibria (see Eq. 27). For this reason we performed numerical simulations of the above bi-stable systems. Even though a systematic quantitative computation of transition times was too demanding, we definitely found that numerical simulations are in qualitative agreement with the picture emerging from the $O(D)$ approximation.

7. Conclusions

We have introduced a simple stochastic dynamics for game theory. This assumes "local" rationality since any player tries to climb the gradient of his utility function. This deterministic process is affected by a stochastic force which represents deviation from rationality in the form of a "heat bath". We focused on particular games with a global interaction which is typical of socio-economic systems: each player's utility depends on his strategy as well as on a global quantity. The stable states of this dynamics coincide with the NE. We studied the Gaussian fluctuations around these NE and established that competitive equilibria are characterized by large fluctuations which grow with the number of players. Large fluctuations imply great inequalities in the distribution of utility among players. Uneven distribution of goods is, unfortunately, very common in the economic world. Our analysis suggests that this is related to the competitive nature of the Nash equilibrium. Players competing for a common resource have broadly distributed utilities whereas players whose strategy is bounded by a direct utility loss (as in the tragedy of the commons with taxes) have more or less the same payoffs. Fluctuations usually decrease the utility of players, but cases where the contrary holds are also possible. Finally we studied the general case in the small noise limit. We found that depending on the particular game, fluctuations can either increase or decrease the dominance of a Pareto dominant state and that new metastable states can occur.

This approach can naturally be extended to games with a discrete strategy space. For these, the Langevin dynamics can be replaced by e.g. Metropolis dynamics where each player tries to minimize his own cost function.

Fluctuations and deviation from rationality are inevitable in the real world. Reducing their strength D_i costs time and money. This suggests a generalization of our work where D_i is considered as a parameter in strategy space. If one then assumes that players have "local" rationality so that the best thing they can do is to climb their utility gradient, one is left with a system where the strategy of each player consists in the choice of the strength of the fluctuations D_i and the rate Γ_i at which they climb the potential. In terms of these parameters $(\vec{D}, \vec{\Gamma})$ we can define a generalized utility function

$$U_i(\vec{D}, \vec{\Gamma}) = \langle u_i(x_i, x_{-i}) | \vec{D}, \vec{\Gamma} \rangle + U_0(D_i, \Gamma_i)\,, \tag{34}$$

where the first term is the average utility discussed in the body of the paper. The second term is related to the cost of achieving a noise reduction to strength D_i and a rate of convergence Γ_i. In general we expect U_0 to be a decreasing function of D_i. Furthermore infinite precision $D_i = 0$ most likely requires an infinite cost. A possible candidate for $-U_0$ is the entropy $U_0(D) \propto \langle \log P(\eta) \rangle = \log \sqrt{D} + C$. In general we found that the average utility decreases with increasing D. In these cases Nash equilibria in strategy space $(\vec{D}, \vec{\Gamma})$ will occur for $D_i > 0$. In the particular cases where we found that $\langle u_i(x_i, x_{-i}) \rangle$ increases with D_i, a large noise strength will be preferred to a more rational behavior. This new approach would provide a more realistic description of real systems of interacting players.

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