PROBABILITY DISTRIBUTIONS GENERATED BY FRACTIONAL DIFFUSION EQUATIONS

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1. Introduction

Non-Gaussian probability distributions are becoming more common as data models, especially in economics, where large fluctuations are expected. In fact, probability distributions with heavy tails are often met in economics and finance, which suggests enlarging the arsenal of possible stochastic models by non-Gaussian processes. This idea dates back to the early sixties after the appearance of a series of papers by Mandelbrot and his associates, who point out the importance of non-Gaussian probability distributions, formerly introduced by Pareto and Lévy, and related scaling properties in analyzing economical and financial variables, as reported in the recent book by Mandelbrot (1997). Some examples of such variables are common stock prices changes, changes in other speculative prices, and interest rate changes. In this respect many works by different authors have recently appeared, see *e.g.* the books by Bouchaud & Potters (1997), Mantegna & Stanley (1998) and the references therein.

It is well known that the fundamental solution (or Green function) of the Cauchy problem for the standard linear diffusion equation gives a Gauss (normal) probability density function (*pdf*) in space. All the moments of this law are finite thanks to its exponential type decay at infinity. In particular, the space variance of the *Green* function is proportional to the first power of time, a noteworthy property that can be understood by means of an unbiased random walk model for the Brownian motion, see *e.g.* Feller (1957). Less well known is that the fundamental solution of the signalling problem for the same diffusion equation, at any position is a unilateral *pdf* in time, known as Lévy's law, using the terminology of Feller (1971). Because of its algebraic decay at infinity as $t^{-3/2}$, this law has all moments of integer order divergent, and consequently its expectation value and variance are infinite.

Both the Gauss and Lévy laws belong to the general class of *stable* probability distributions, which are characterized by an index α $(0 < \alpha \leq 2)$, called index of stability or characteristic exponent. In particular, the index of the Gauss law is 2, whereas that of the Lévy law is $1/2$.

In this paper we consider two different generalizations of the diffusion equation by means of fractional calculus, which allows us to replace either the first time derivative or the second space derivative by a suitable fractional derivative. Correspondingly, the generalized equation will be referred to as the *time-fractional* diffusion equation or the *symmetric, space-fractional* diffusion equation. Here we show how the fundamental solutions of this equation for the Cauchy and Signalling problems provide probability density functions related to certain stable distributions, thereby providing a natural generalization of what occurs for the standard diffusion equation.

The plan of the paper is as follows. First of all, for the sake of convenience and completeness, we provide the essential notions of the Riemann-Liouville fractional calculus and Lévy stable probability distributions in Appendix A and B, respectively.

In Sec. 2 we recall the basic results for the standard diffusion equation concerning the fundamental solutions of the Cauchy and signalling problems. In particular, we provide the derivation of these solutions by the Fourier and Laplace transforms and their interpretations in terms of Gauss and Lévy stable *pdf's* respectively.

In Sec. 3 we consider the time-fractional diffusion equation and we formulate for it the basic Cauchy and Signalling problems to be treated in the subsequent two sections. Here we adopt the Riemann-Liouville approach to fractional calculus, and the related definition for the Caputo time-fractional derivative of a causal function of time.

In Sec. 4 we solve the Cauchy problem for the time-fractional diffusion equation by Fourier transformation and we derive the corresponding fundamental solution in terms of a special function of the Wright type in the similarity variable. In this case the solution can be interpreted as a noteworthy symmetric *pdf* in space with all moments finite, evolving in time.

In particular, its space variance turns out to be proportional to a power of time equal to the order of the time-fractional derivative.

In Sec. 5 we derive the fundamental solution for the signalling problem of the time-fractional diffusion equation by Laplace transformation. In this case the solution, still expressed in terms of a special function of the Wright type, can be interpreted as a unilateral stable *pdf* in time, depending on position, with index of stability given by one half of the order of the time-fractional derivative.

In Sec. 6 we consider the *symmetric, space-fractional* diffusion equation. Here we adopt the Riesz approach to fractional calculus, and the related definition for the symmetric space-fractional derivative of a function of a single space variable. Here we treat the Cauchy problem by Fourier transformation and we derive the series representation of the corresponding Green function. In this case the fundamental solution is interpreted in terms of a symmetric stable *pdf* in space, evolving in time, with the index of stability given by the order of the space-fractional derivative. To approximate such evolution we propose a random walk model, discrete in space and time, which is based on the Grünwald-Letnikov approximation of the fractional derivative.

Finally, Sec. 7 is devoted to conclusions and remarks on related works.

2. The Standard Diffusion Equation

By standard diffusion equation we mean the linear partial differential equation

$$\frac{\partial}{\partial t}\, u(x,t) = \mathcal{D}\, \frac{\partial^2}{\partial x^2}\, u(x,t)\,, \qquad u = u(x,t)\,, \tag{2.1}$$

where $\mathcal{D}$ denotes a positive constant with dimensions $L^2\,T^{-1}$, x and t are the space-time variables, and $u = u(x,t)$ is the field variable, which is assumed to be a *causal* function of time, *i.e.* vanishing for $t < 0$.

The typical physical phenomenon related to such an equation is heat conduction in a thin solid rod extended along x, so the field variable u is the temperature.

In order to guarantee the existence and the uniqueness of the solution, we must equip (1.1) with suitable boundary conditions. The basic boundary-value problems for diffusion are the so-called *Cauchy* and *Signalling* problems. In the *Cauchy* problem, which concerns the space-time domain $-\infty < x < +\infty\,,\ t \ge 0$, the data are assigned at $t = 0^+$ on the whole space axis (initial data). In the *Signalling* problem, which concerns the space-time domain $x \ge 0\,,\ t \ge 0$, the data are assigned both at $t = 0^+$ on the semi-infinite space axis $x > 0$ (initial data) and at $x = 0^+$ on the semi-infinite time axis $t > 0$ (boundary data); here, as usual, the initial data are assumed to be vanishing.

Denoting by $g(x)$ and $h(t)$ two given, sufficiently well-behaved functions, the basic problems are thus formulated as follows:

a) Cauchy problem

$$u(x,0^+) = g(x)\,, \quad -\infty < x < +\infty\,; \quad u(\mp\infty,t) = 0\,, \quad t > 0\,; \qquad (2.2a)$$

b) Signalling problem

$$u(x,0^+) = 0\,, \quad x > 0\,; \quad u(0^+,t) = h(t)\,, \quad u(+\infty,t) = 0\,, \quad t > 0\,. \qquad (2.2b)$$

Hereafter, for both problems, we derive the classical results which will be generalized for the case of the fractional diffusion equation in the subsequent sections.

Let us begin with the *Cauchy* problem. It is well known that this initial value problem can be easily solved by Fourier transform and its fundamental solution can be interpreted as a Gaussian *pdf* in x. Adopting the notation $g(x) \div \hat{g}(\kappa)$ with $\kappa \in \mathbf{R}$ and

$$\hat{g}(\kappa) = \mathcal{F}\left[g(x)\right] = \int_{-\infty}^{+\infty} \mathrm{e}^{\,+i\kappa x}\, g(x)\, dx\,,$$

$$g(x) = \mathcal{F}^{-1}\left[\hat{g}(\kappa)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\,-i\kappa x}\, \hat{g}(\kappa)\, d\kappa\,,$$

the transformed solution satisfies the ordinary first order differential equation

$$\left(\frac{d}{dt} + \kappa^2\,\mathcal{D}\right)\,\hat{u}(\kappa,t) = 0\,, \quad \hat{u}(\kappa,0^+) = \hat{g}(\kappa)\,, \qquad (2.3)$$

and consequently it turns out to be

$$\hat{u}(\kappa,t) = \hat{g}(\kappa)\,\mathrm{e}^{\,-\kappa^2\,\mathcal{D}\,t}\,. \qquad (2.4)$$

Then, introducing

$$\mathcal{G}_c^d(x,t) \div \widehat{\mathcal{G}_c^d}(\kappa,t) = \mathrm{e}^{-\kappa^2 \mathcal{D}\, t}\,, \tag{2.5}$$

where the upper index d refers to (standard) diffusion, the required solution, obtained by inversion of (2.4), can be expressed in terms of the space convolution $u(x,t) = \int_{-\infty}^{+\infty} \mathcal{G}_c^d(\xi,t)\, g(x-\xi)\, d\xi$, where

$$\mathcal{G}_c^d(x,t) = \frac{1}{2\sqrt{\pi\, \mathcal{D}}}\, t^{-1/2}\, \mathrm{e}^{-x^2/(4\, \mathcal{D}\, t)}\,. \tag{2.6}$$

Here $\mathcal{G}_c^d(x,t)$ represents the fundamental solution (or *Green* function) of the *Cauchy* problem, since it corresponds to $g(x) = \delta(x)$. It turns out to be an even and normalized function in x , *i.e.* $\mathcal{G}_c^d(x,t) = \mathcal{G}_c^d(|x|,t)$ and $\int_{-\infty}^{+\infty} \mathcal{G}_c^d(x,t)\, dx = 1$. We also note the identity

$$|x|\, \mathcal{G}_c^d(|x|,t) = \frac{\zeta}{2}\, M^d(\zeta)\,, \tag{2.7}$$

where $\zeta = |x|/(\sqrt{\mathcal{D}}\, t^{1/2})$ is the well-known *similarity* variable and

$$M^d(\zeta) = \frac{1}{\sqrt{\pi}}\, \mathrm{e}^{-\zeta^2/4}\,. \tag{2.8}$$

We note that $M^d(\zeta)$ satisfies the normalization condition $\int_0^\infty M^d(\zeta)\, d\zeta = 1$.

The interpretation of the *Green* function (2.6) in probability theory is straightforward since we easily recognize

$$\mathcal{G}_c^d(x,t) = p_G(x;\sigma) := \frac{1}{\sqrt{2\pi}\, \sigma}\, \mathrm{e}^{-x^2/(2\sigma^2)}\,, \quad \sigma^2 = 2\, \mathcal{D}\, t\,, \tag{2.9}$$

where $p_G(x;\sigma)$ denotes the well-known *Gauss* or *normal pdf* spread out over all real x (the space variable), whose moment of the second order, the *variance*, is σ^2 . The associated cumulative distribution function (*cdf*) is known to be

$$\mathcal{P}_G(x;\sigma) := \int_{-\infty}^{x} p_G(x';\sigma)\, dx' = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}\, \sigma}\right)\right]\,, \tag{2.10}$$

where $\mathrm{erf}\,(z) := (2/\sqrt{\pi}) \int_0^z \exp\,(-u^2)\, du$ denotes the error function. Furthermore, the moments of even order of the *Gauss pdf* turn out to be $\int_{-\infty}^{+\infty} x^{2n}\, p_G(x;\sigma)\, dx = (2n-1)!!\, \sigma^{2n}$, so

$$\int_{-\infty}^{+\infty} x^{2n}\, \mathcal{G}_c^d(x,t)\, dx = (2n-1)!!\, (2\mathcal{D}\, t)^n\,, \quad n = 1, 2, \ldots\,. \tag{2.11}$$

Let us now consider the *Signalling* problem. This initial-boundary value problem can be easily solved by making use of the Laplace transform. Adopting the notation $h(t) \div \tilde{h}(s)$ with $s \in \mathbf{C}$ and

$$\tilde{h}(s) = \mathcal{L}\left[h(t)\right] = \int_0^\infty \mathrm{e}^{-st}\, h(t)\, dt\,,$$

$$h(t) = \mathcal{L}^{-1}\left[\tilde{h}(t)\right] = \frac{1}{2\pi i}\int_{Br} \mathrm{e}^{st}\, \tilde{h}(s)\, ds\,,$$

where Br denotes the Bromwich path, the transformed solution of the diffusion equation satisfies the ordinary second order differential equation

$$\left(\frac{d^2}{dx^2} + \frac{s}{\mathcal{D}}\right)\tilde{u}(x,s) = 0\,, \quad \tilde{u}(0^+,s) = \tilde{h}(s)\,, \quad \tilde{u}(+\infty,s) = 0\,. \tag{2.12}$$

and consequently it turns out to be

$$\tilde{u}(x,s) = \tilde{h}(s)\,\mathrm{e}^{-(x/\sqrt{\mathcal{D}})\,s^{1/2}}\,. \tag{2.13}$$

Then introducing

$$\mathcal{G}_s^d(x,t) \div \widetilde{\mathcal{G}}_s^d(x,s) = \mathrm{e}^{-(x/\sqrt{\mathcal{D}})\,s^{1/2}}\,, \tag{2.14}$$

the required solution, obtained by inversion of (2.13), can be expressed in terms of the time convolution $u(x,t) = \int_0^t \mathcal{G}_s^d(x,\tau)\, h(t-\tau)\, d\tau$, where

$$\mathcal{G}_s^d(x,t) = \frac{x}{2\sqrt{\pi\,\mathcal{D}}}\, t^{-3/2}\, \mathrm{e}^{-x^2/(4\,\mathcal{D}\,t)}\,. \tag{2.15}$$

Here $\mathcal{G}_s^d(x,t)$ represents the fundamental solution (or *Green* function) of the *Signalling* problem, since it corresponds to $h(t) = \delta(t)$. We note that

$$\mathcal{G}_s^d(x,t) = p_L(t;\mu) := \frac{\sqrt{\mu}}{\sqrt{2\pi}\, t^{3/2}}\, \mathrm{e}^{-\mu/(2t)}\,, \quad t \ge 0\,, \quad \mu = \frac{x^2}{2\,\mathcal{D}}\,, \tag{2.16}$$

where $p_L(t;\mu)$ denotes the *one-sided Lévy pdf* spread out over all non negative t (the time variable). The associated *cdf* is, see *e.g.* Feller (1971) and Prüsse (1993),

$$\mathcal{P}_L(t;\mu) := \int_0^t p_L(t';\mu)\, dt' = \mathrm{erfc}\left(\sqrt{\frac{\mu}{2t}}\right) = \mathrm{erfc}\left(\frac{x}{2\sqrt{\mathcal{D}\,t}}\right)\,, \tag{2.17}$$

where $\mathrm{erfc}\,(z) := 1 - \mathrm{erf}\,(z)$ denotes the complenatary error function.

All integer order moments of the *Lévy pdf* are infinite, since it decays at infinity as $t^{-3/2}$. However, we note that the absolute moments of real order ν are finite only if $0 \le \nu < 1/2$. In particular, for this *pdf* the mean is infinite, so we can take the *median* instead of the expectation value. From $\mathcal{P}_L(t_{med};\mu) = 1/2$, it turns out that $t_{med} \approx 2\mu$, since the complementary error function takes on the value 1/2 as its argument at approximatively 1/2.

We note that in the common domain $x > 0$, $t > 0$ the Green functions of the two basic problems satisfy the identity

$$x\,\mathcal{G}_c^d(x,t) = t\,\mathcal{G}_s^d(x,t)\,, \tag{2.18}$$

that we refer to as the *reciprocity relation* between the two fundamental solutions of the diffusion equation. Furthermore, in view of (2.7) and (2.18) we recognize the role of the function of the similarity variable, $M^d(\zeta)$, in providing the two fundamental solutions; we shall refer to it as the *normalized auxiliary function* of the diffusion equation for both the *Cauchy* and *Signalling* problems.

3. The Time-Fractional Diffusion Equation

By *time-fractional diffusion* equation we mean the linear evolution equation obtained from the classical diffusion equation by replacing the first-order time derivative by a fractional derivative (in the *Caputo* sense) of order α with $0 < \alpha \le 2$. In our notation this reads

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \mathcal{D}\,\frac{\partial^2 u}{\partial x^2}\,, \qquad u = u(x,t)\,, \quad 0 < \alpha \le 2\,, \tag{3.1}$$

where $\mathcal{D}$ denotes a positive constant with dimensions $L^2\,T^{-\alpha}$. From Appendix A we recall the definition of the *Caputo* fractional derivative of order $\alpha > 0$ for a (sufficiently well-behaved) *causal* function $f(t)$, see (A.9),

$$D_*^\alpha\, f(t) := \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha}\, f^{(m)}(\tau)\,d\tau\,, \tag{3.2}$$

where $m = 1, 2, \ldots$, and $0 \le m-1 < \alpha \le m$. According to (3.2) we thus need to distinguish the cases $0 < \alpha \le 1$ and $1 < \alpha \le 2$. In the latter case, (3.1) may be seen as a sort of interpolation between the standard diffusion equation and the standard wave equation. Introducing

$$\Phi_\lambda(t) := \frac{t_+^{\lambda-1}}{\Gamma(\lambda)}\,, \quad \lambda > 0\,, \tag{3.3}$$

where the suffix + means that the function is vanishing for $t < 0$, we easily recognize that equation (3.1) assumes the explicit forms :
if $0 < \alpha \le 1$,

$$\Phi_{1-\alpha}(t)\, *\, \frac{\partial u}{\partial t} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\left(\frac{\partial u}{\partial \tau}\right) d\tau = \mathcal{D}\,\frac{\partial^2 u}{\partial x^2}\,; \tag{3.4}$$

if $1 < \alpha \le 2$,

$$\Phi_{2-\alpha}(t)\, *\, \frac{\partial^2 u}{\partial t^2} = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-\tau)^{1-\alpha}\left(\frac{\partial^2 u}{\partial \tau^2}\right) d\tau = \mathcal{D}\,\frac{\partial^2 u}{\partial x^2}\,. \tag{3.5}$$

Extending the classical analysis for the standard diffusion equation (2.1) to the above integro-differential equations (3.4-5), the *Cauchy* and *Signalling* problems are thus formulated as in equations (2.2), i.e.

a) Cauchy problem

$$u(x,0^+) = g(x)\,, \quad -\infty < x < +\infty\,; \quad u(\mp\infty,t) = 0\,, \quad t > 0\,; \qquad (3.6a)$$

b) Signalling problem

$$u(x,0^+) = 0\,, \quad x > 0\,; \quad u(0^+,t) = h(t)\,, \quad u(+\infty,t) = 0\,, \quad t > 0\,. \; (3.6b)$$

However, if $1 < \alpha \le 2$, the presence in (3.5) of the second order time derivative of the field variable requires us to specify the initial value of the first order time derivative $u_t(x,0^+)$, since in this case two linearly independent solutions are to be determined. To ensure the continuous dependence of our solution on the parameter α in the transition from $\alpha = 1^-$ to $\alpha = 1^+$, we assume $u_t(x,0^+) = 0$.

We recognize that our fractional diffusion equation (3.1), when subject to conditions (3.6), is equivalent to the integro-differential equation

$$u(x,t) = g(x) + \frac{\mathcal{D}}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left(\frac{\partial^2 u}{\partial x^2}\right) d\tau\,, \qquad (3.7)$$

where $0 < \alpha \le 2$. Such integro-differential equations have been investigated by several authors, including Schneider & Wyss (1989), Fujita (1990), Prüsse (1993) and Engler (1997).

In view of our subsequent analysis we find it convenient to put

$$\nu = \frac{\alpha}{2}\,, \quad 0 < \nu < 1\,. \qquad (3.8)$$

In fact the analysis of the *time-fractional* diffusion equation turns out to be easier if we adopt as key parameter one half of the order of the *time-fractional* derivative. Henceforth, we shall invest the symbol α with other relevant meanings, as the index of stability of a stable probability distribution or the order of the space derivative in the *space-fractional* diffusion equation.

Hereafter, we agree to insert the parameter ν in the field variable, *i.e.* $u = u(x,t;\nu)$. By denoting the Green functions of the *Cauchy* and *Signalling* problems by $\mathcal{G}_c(x,t;\nu)$ and $\mathcal{G}_s(x,t;\nu)$, respectively, the solutions of the two basic problems are obtained by a space or time convolution, $u(x,t;\nu) = \int_{-\infty}^{+\infty} \mathcal{G}_c(\xi,t;\nu)\, g(x-\xi)\, d\xi$, $u(x,t;\nu) = \int_0^t \mathcal{G}_s(x,\tau;\nu)\, h(t-\tau)\, d\tau$, respectively. It should be noted that $\mathcal{G}_c(x,t;\nu) = \mathcal{G}_c(|x|,t;\nu)$, since the Green function turns out to be an even function of x.

In the following two sections we shall compute the two fundamental solutions with the same techniques (based on Fourier and Laplace transforms) used for the standard diffusion equation and we shall provide their interpretation in terms of probability distributions. Most of the presented results are based on the papers by Mainardi (1994), (1995), (1996), (1997) and by Mainardi & Tomirotti (1995), (1997).

4. The Cauchy Problem for the Time-Fractional Diffusion Equation

For the fractional diffusion equation (3.1) subject to (3.6a) Fourier transformation leads to the ordinary differential equation of order $\alpha = 2\nu$,

$$\left(\frac{d^{2\nu}}{dt^{2\nu}} + \kappa^2\,\mathcal{D}\right)\,\hat{u}(\kappa,t;\nu) = 0\,, \quad \hat{u}(\kappa,0^+;\nu) = \hat{g}(\kappa)\,, \tag{4.1}$$

Using the results of Appendix A, see (A.22-30), the transformed solution is

$$\hat{u}(\kappa,t;\nu) = \hat{g}(\kappa)\,E_{2\nu}\left(-\kappa^2\,\mathcal{D}\,t^{2\nu}\right)\,, \tag{4.2}$$

where $E_{2\nu}(\cdot)$ denotes the *Mittag-Leffler* function of order 2ν, and consequently for the *Green* function we have

$$\mathcal{G}_c(x,t;\nu) = \mathcal{G}_c(|x|,t;\nu) \div \hat{\mathcal{G}}_c(k,t;\nu) = E_{2\nu}\left(-\kappa^2\mathcal{D}\,t^{2\nu}\right)\,. \tag{4.3}$$

Since the *Green* function is a real and even function of x, its (exponential) Fourier transform can be expressed in terms of the *cosine* Fourier transform and thus is related to its spatial Laplace transform as follows:

$$\begin{aligned}\hat{\mathcal{G}}_c(k,t;\nu) &= 2\int_0^\infty \mathcal{G}_c(x,t;\nu)\,\cos\kappa x\,dx = \\ &\quad \tilde{\mathcal{G}}_c(s,t;\nu)\Big|_{s=+ik} + \tilde{\mathcal{G}}_c(s,t;\nu)\Big|_{s=-ik}\,.\end{aligned} \tag{4.4}$$

Indeed, a split also occurs in (4.3) according to the *duplication formula* for the *Mittag-Leffler* function, see (A.26),

$$\begin{aligned}\hat{\mathcal{G}}_c(k,t;\nu) &= E_{2\nu}(-\kappa^2\,\mathcal{D}\,t^{2\nu}) = \\ &\quad [E_\nu(+i\kappa\,\sqrt{\mathcal{D}}\,t^\nu) + E_\nu(-i\kappa\,\sqrt{\mathcal{D}}\,t^\nu)]/2\,.\end{aligned} \tag{4.5}$$

When $\nu \neq 1/2$ the inversion of the Fourier transform in (4.5) cannot be obtained by using a standard table of Fourier transform pairs; however, for any $\nu \in (0,1)$ such an inversion can be achieved by turning to the

Laplace transform pair (A.37) with $r = |x|$, and $s = \pm i\kappa$. In fact, taking into account the scaling property of the Laplace transform, we obtain from (4.5) and (A.37)

$$\mathcal{G}_c(|x|, t; \nu) = \frac{1}{2\sqrt{\mathcal{D}}\, t^\nu}\, M\left(\frac{|x|}{\sqrt{\mathcal{D}} t^\nu}; \nu\right), \tag{4.6}$$

where $M(\zeta;\nu)$ is the special function of Wright type, defined by (A.31-33), and

$$\zeta = \frac{|x|}{\sqrt{\mathcal{D}} t^\nu}, \tag{4.7}$$

the *similarity* variable. We note the identity

$$|x|\, \mathcal{G}_c(|x|, t; \nu) = \frac{\zeta}{2}\, M(\zeta;\nu), \tag{4.8}$$

which generalizes to the time-fractional diffusion equation the identity (2.7) of the standard diffusion equation. Since $\int_0^\infty M(\zeta;\nu)\, d\zeta = 1$, see (A.40), the function $M(\zeta;\nu)$ is the *normalized auxiliary* function of the fractional diffusion equation.

We note that for the time-fractional diffusion equation the fundamental solution of the *Cauchy* problem is still a bilateral symmetric *pdf* in x (with two branches, for $x > 0$ and $x < 0$, obtained one from the other by reflection), but is no longer of the Gaussian type if $\nu \neq 1/2$. In fact, for large $|x|$ each branch exhibits an exponential decay in the "stretched" variable $|x|^{1/(1-\nu)}$ as can be derived from the asymptotic representation (A.36) of the *auxiliary* function $M(\cdot;\nu)$. In fact, by using (4.7-8) and (A.36), we obtain

$$\mathcal{G}_c(x, t; \nu) \sim a_*(t)\, |x|^{(\nu-1/2)/(1-\nu)} \exp\left[-b_*(t)|x|^{1/(1-\nu)}\right], \tag{4.9}$$

as $|x| \to \infty$, where $a_*(t)$ and $b_*(t)$ are certain positive functions of time

Furthermore, the exponential decay in x provided by (4.9) ensures that all the absolute moments of positive order of $\mathcal{G}_c(x, t; \nu)$ are finite. In particular, using (4.8) and (A.39) it turns out that the moments (of even order) are

$$\int_{-\infty}^{+\infty} x^{2n}\, \mathcal{G}_c(x, t; \nu)\, dx = \frac{\Gamma(2n+1)}{\Gamma(2\nu n+1)}\, (\mathcal{D}t^{2\nu})^n, \quad n = 0, 1, 2, \ldots \tag{4.10}$$

Formula (4.10) provides a generalization of the corresponding formula (2.11) valid for the standard diffusion equation, $\nu = 1/2$. Furthermore, we recognize that the variance associated with the *pdf* is now proportional to $\mathcal{D}t^{2\nu}$, which for $\nu \neq 1/2$ implies the phenomenon of *anomalous* diffusion. According to standard terminology in statistical mechanics, the anomalous diffusion is said to be slow if $0 < \nu < 1/2$ and fast if $1/2 < \nu < 1$.

In Figure 1, as an example, we compare with $|x|$, at fixed t, the fundamental solutions of the *Cauchy* problem with different ν ($\nu = 1/4\,,\ 1/2\,, 3/4$). We consider the range $0 \le |x| \le 4$ and assume $\mathcal{D} = t = 1$.

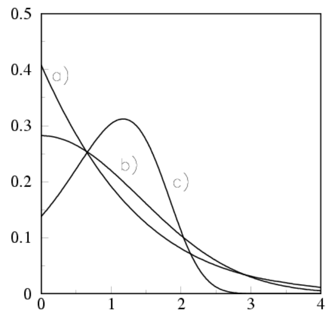

Figure 1. The Cauchy problem for the time-fractional diffusion equation. The fundamental solutions against $|x|$ with a) $\nu = 1/4$, b) $\nu = 1/2$, c) $\nu = 3/4$.

We note the different behaviour of the *pdf* in the case of slow diffusion ($\nu = 1/4$) and fast diffusion ($\nu = 3/4$) compared to the Gaussian behaviour of standard diffusion ($\nu = 1/2$). In the limiting cases $\nu = 0$ and $\nu = 1$ we have

$$\mathcal{G}_c(x,t;0) = \frac{\mathrm{e}^{-|x|}}{2}\,, \quad \mathcal{G}_c(x,t;1) = \frac{\delta(x-\sqrt{\mathcal{D}}\,t) + \delta(x+\sqrt{\mathcal{D}}\,t)}{2}\,. \qquad (4.11)$$

We also recognize from appendix B that for $1/2 \le \nu < 1$ any branch of the fundamental solution is proportional to the corresponding *positive* branch of an extremal stable *pdf* with index of stability $\alpha = 1/\nu$, which exhibits an exponential decay at infinity. In fact, applying (B.29) with $\alpha = 1/\nu$ and $y = \zeta = |x|/(\sqrt{\mathcal{D}}t^\nu)$, from (4.7-8) we obtain

$$\begin{aligned} \mathcal{G}_c(|x|,t;\nu) \;=\; & \frac{1}{2\sqrt{\mathcal{D}}\,t^\nu}\cdot\frac{1}{\nu}\,q_{1/\nu}\left[|x|/(\sqrt{\mathcal{D}}\,t^\nu)\,;\,-(2-1/\nu)\right] = \\ & \frac{1}{2\nu}\cdot p_{1/\nu}\,(|x|;\,+1,1,0)\,, \qquad 1 < 1/\nu \le 2\,. \end{aligned} \qquad (4.12)$$

We also note that the stable distribution in (4.12) satisfies the condition

$$\int_0^{+\infty} p_{1/\nu}\,(x;\,+1,1,0)\,dx = \nu\,, \qquad 1 < 1/\nu \le 2\,. \qquad (4.13)$$

5. The Signalling Problem for the Time-Fractional Diffusion Equation

For the fractional diffusion equation (3.1) subject to (3.6b) the application of the Laplace transform leads to the ordinary second order differential equation

$$\left(\frac{d^2}{dx^2} + \frac{s^{2\nu}}{\mathcal{D}}\right) \tilde{u}(x,s;\nu)\,, \quad \tilde{u}(0^+,s;\nu) = \tilde{h}(s)\,, \quad \tilde{u}(+\infty,s;\nu) = 0\,. \tag{5.1}$$

Thus, the transformed solution reads

$$\tilde{u}(x,s;\nu) = \tilde{h}(s)\,\mathrm{e}^{-(x/\sqrt{\mathcal{D}})\,s^\nu}\,, \tag{5.2}$$

and for the *Green* function we have

$$\mathcal{G}_s(x,t;\nu) \div \tilde{\mathcal{G}}_s(x,s;\nu) = \mathrm{e}^{-(x/\sqrt{\mathcal{D}})\,s^\nu}\,. \tag{5.3}$$

When $\nu \neq 1/2$ the inversion of this Laplace transformation cannot be obtained by looking in a standard table of Laplace transform pairs. Here too we turn to a Laplace transform pair related to the Wright-type function $M(\zeta;\nu)$. In fact, using (A.40) with $r = t$, and taking into account the scaling property of the Laplace transform, we obtain

$$\mathcal{G}_s(x,t;\nu) = \nu\,\frac{x}{\sqrt{\mathcal{D}}\,t^{1+\nu}}\,M\left(\frac{x}{\sqrt{\mathcal{D}}\,t^\nu};\nu\right)\,. \tag{5.4}$$

Introducing the *similarity* variable $\zeta = x/(\sqrt{\mathcal{D}}t^\nu)$, we recognize the identity

$$t\,\mathcal{G}_s(x,t;\nu) = \nu\,\zeta\,M(\zeta;\nu)\,, \tag{5.5}$$

which is the counterpart to the *Signalling* problem of identity (4.8) valid for the *Cauchy* problem.

Comparing (5.5) with (4.8) we obtain the *reciprocity relation* between the two fundamental solutions of the *time-fractional* diffusion equation, in the common domain $x > 0$, $t > 0$,

$$2\nu\,x\,\mathcal{G}_c(x,t;\nu) = t\,\mathcal{G}_s(x,t;\nu)\,. \tag{5.6}$$

The interpretation of $\mathcal{G}_s(x,t;\nu)$ as a one-sided *stable pdf* in time is straightforward: we need to apply (B.28), with index of stability $\alpha = \nu$ and variable $y = \zeta^{-1/\nu} = t\,(\sqrt{\mathcal{D}}/x)^{1/\nu}$, in (5.5). We obtain

$$\mathcal{G}_s(x,t;\nu) = \left(\frac{\sqrt{\mathcal{D}}}{x}\right)^{1/\nu} q_\nu\left[t\left(\frac{\sqrt{\mathcal{D}}}{x}\right)^{1/\nu};-\nu\right] = p_\nu\,(t;\,-1,1,0)\,. \tag{5.7}$$

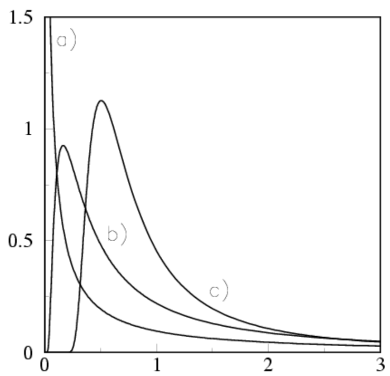

Figure 2. The Signalling problem for the time-fractional diffusion equation. The fundamental solutions against t with a) $\nu = 1/4$, b) $\nu = 1/2$, c) $\nu = 3/4$.

In Figure 2, for example, we compare with t, at fixed x, the fundamental solutions of the *Signalling* problem with different ν ($\nu = 1/4,\ 1/2, 3/4$). We consider the range $0 \le t \le 3$ and assume $\mathcal{D} = x = 1$.

We note the different behaviour of the *pdf* in the case of slow diffusion ($\nu = 1/4$) and fast diffusion ($\nu = 3/4$) compared to the *Lévy pdf* for standard diffusion ($\nu = 1/2$). In the limiting cases $\nu = 0\,,\,1$, we have

$$\mathcal{G}_s(x,t;0) = \delta(t)\,, \qquad \mathcal{G}_s(x,t;1) = \delta(t - x/\sqrt{\mathcal{D}})\,. \tag{5.8}$$

6. The Cauchy Problem for the Symmetric Space-Fractional Diffusion Equation

The *symmetric space-fractional diffusion* equation is obtained from the classical diffusion equation by replacing the second-order space derivative with a symmetric space-fractional derivative (explained below) of order α with $0 < \alpha \le 2$. In our notation we write this equation as

$$\frac{\partial u}{\partial t} = \mathcal{D}\,\frac{\partial^\alpha u}{\partial |x|^\alpha}\,, \quad u = u(x,t;\alpha)\,,\ x \in \mathbf{R}\,,\ t \in \mathbf{R}_0^+\,, \quad 0 < \alpha \le 2\,, \tag{6.1}$$

where $\mathcal{D}$ is a positive coefficient with dimensions $L^\alpha\,T^{-1}$. The fundamental solution of the *Cauchy* problem, $\mathcal{G}_c(x,t;\alpha)$ is the solution of (6.1), subject to the initial condition $u(x,0^+;\alpha) = \delta(x)$.

The *symmetric space-fractional* derivative of any order $\alpha > 0$ of a sufficiently well-behaved function $\phi(x)$, $x \in \mathbf{R}$, may be defined as the pseudo-differential operator characterized in its Fourier representation by

$$\frac{d^\alpha}{d|x|^\alpha}\,\phi(x) \;\div\; -|\kappa|^\alpha\,\hat{\phi}(\kappa)\,, \quad x\,,\,k \in \mathbf{R}\,, \quad \alpha > 0\,. \tag{6.2}$$

According to standard terminology, $-|\kappa|^\alpha$ is referred to as the symbol of our pseudo-differential operator, the *symmetric space-fractional* derivative, of order α. Here, we have adopted the notation introduced by Zaslavski, see *e.g.* Saichev & Zaslavski (1997).

In order to properly introduce this kind of fractional derivative we need to consider a particular approach to fractional calculus, different from the *Riemann-Liouville* one already treated in Appendix A. This approach is based on the so-called *Riesz* potentials (or integrals)which we prefer to consider later.

First, let us see how things become highly transparent by using an heuristic argument, originally due to Feller (1952). The idea is to start from the positive definite differential operator

$$A := -\frac{d^2}{dx^2} \;\div\; \kappa^2 = |\kappa|^2\,, \tag{6.3}$$

whose symbol is $|\kappa|^2$, and form positive powers of this operator as pseudo-differential operators by their action in the Fourier-image space, i.e.

$$A^{\alpha/2} := \left(-\frac{d^2}{dx^2}\right)^{\alpha/2} \;\div\; \left(|\kappa|^2\right)^{\alpha/2} = |\kappa|^\alpha \quad \alpha > 0\,. \tag{6.4}$$

Thus, the operator $-A^{\alpha/2}$ can be interpreted as the required fractional derivative, i.e.

$$A^{\alpha/2} \equiv -\frac{d^\alpha}{d|x|^\alpha}\,, \qquad \alpha > 0\,. \tag{6.5}$$

We note that the operator just defined must not be confused with a power of the first order differential operator $\frac{d}{dx}$ for which the symbol is $-i\kappa$.

After the above considerations it is straightforward to obtain the Fourier image of the *Green* function of the *Cauchy* problem for the *space-fractional* diffusion equation. In fact, applying the Fourier transform to equation (6.1), subject to the initial condition $u(x,0^+;\alpha) = \delta(x)$, and accounting for (6.2), we obtain

$$\mathcal{G}_c(x,t;\alpha) = \mathcal{G}_c(|x|,t;\alpha) \div \hat{\mathcal{G}}_c(k,t;\alpha) = \mathrm{e}^{-\mathcal{D}\,t\,|\kappa|^\alpha}\,, \quad 0 < \alpha \le 2\,. \tag{6.6}$$

We easily recognize that the Fourier transform of the Green function corresponds to the canonic form of a *symmetric* stable distribution with index of stability α and scaling factor $\gamma = (\mathcal{D}t)^{1/\alpha}$, see (B.8). Therefore we have

$$\mathcal{G}_c(x,t;\alpha) = p_\alpha(x;0,\gamma,0)\,, \quad \gamma = (\mathcal{D}t)^{\,1/\alpha}\,. \tag{6.7}$$

For $\alpha = 1$ and $\alpha = 2$ we easily obtain the explicit expressions of the corresponding Green functions since in these cases they correspond to the *Cauchy* and *Gauss* distributions,

$$\mathcal{G}_c(x,t;1) = \frac{1}{\pi}\,\frac{\mathcal{D}\,t}{x^2 + (\mathcal{D}\,t)^2}\,, \tag{6.8}$$

see (B.5), and

$$\mathcal{G}_c(x,t;2)) = \frac{1}{2\sqrt{\pi\,\mathcal{D}\,t}}\,\mathrm{e}^{-x^2/(4\,\mathcal{D}\,t)}\,, \tag{6.9}$$

in agreement with (2.6).

We easily recognize that

$$\eta := \frac{|x|}{(\mathcal{D}\,t)^{1/\alpha}} \tag{6.10}$$

is the *similarity* variable for the *space-fractional* diffusion equation, in terms of which we can express the *Green* function for any $\alpha \in (0,2]$. Indeed, we recognize that

$$\mathcal{G}_c(x,t;\alpha) = \frac{1}{(\mathcal{D}\,t)^{1/\alpha}}\,q_\alpha(\eta;0)\,, \tag{6.11}$$

where $q_\alpha(\eta;0)$ denotes the *symmetric* stable distribution of order α with Feller-type characteristic function, see (B.14-15). Now we can express the Green function using the Feller series expansions (B.21-22) with $\theta = 0$. We obtain:

for $0 < \alpha < 1$

$$q_\alpha(\eta;0) = -\frac{1}{\pi\,\eta}\sum_{n=1}^{\infty}\frac{\Gamma(n\alpha+1)}{n!}\,\sin\left(n\frac{\alpha\pi}{2}\right)\left(-\eta^{\,-\alpha}\right)^n\,, \tag{6.12a}$$

for $1 < \alpha \le 2$

$$q_\alpha(\eta;0) = \frac{1}{\pi\,\alpha}\sum_{m=0}^{\infty}(-1)^m\,\frac{\Gamma[(2m+1)/\alpha]}{(2m)!}\,\eta^{2m}\,. \tag{6.12b}$$

In the limiting case $\alpha = 1$, the above series reduce to geometrical series and are therefore no longer convergent in all of $\mathbf{C}$. In particular, they represent the expansions of the function $q_1(\eta;0) = 1/[\pi(1+\eta^2)]$, convergent for $\eta > 1$ and $0 < \eta < 1$, respectively.

We also note that for any $\alpha \in (0,2]$ the functions $q_\alpha(\eta;0)$ take on the value $q_\alpha(0;0) = \Gamma(1/\alpha)/(\pi\,\alpha)$, at the origin and at the queues, excluding the Gaussian case $\alpha = 2$, they exhibit the *algebraic* asymptotic behaviour at the tails, as $\eta \to \infty$,

$$q_\alpha(\eta;0) \sim \frac{1}{\pi}\,\Gamma(\alpha+1)\,\sin\left(\alpha\frac{\pi}{2}\right)\,\eta^{-(\alpha+1)}\,, \quad 0<\alpha<2\,. \tag{6.13}$$

In Figure 3, as an example, we compare with x, at fixed t, the fundamental solutions of the *Cauchy* problem with different α ($\alpha = 1/2\,, 1\,, 3/2\,, 2$). We consider the range $-6 \le x \le +6$ and assume $\mathcal{D} = t = 1$.

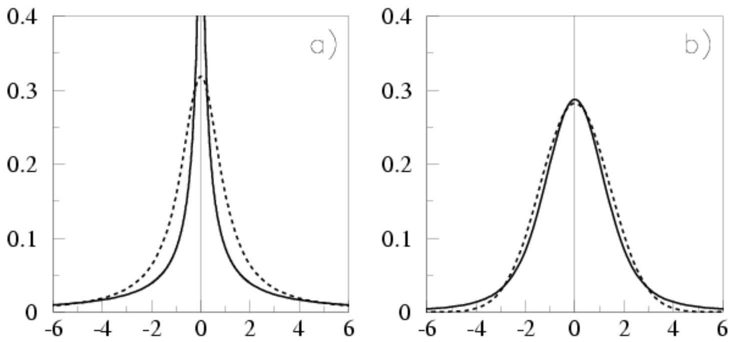

Figure 3. The Cauchy problem for the symmetric space-fractional diffusion equation. The fundamental solutions against x : plate a) $\alpha = 1/2$ (continuous line), $\alpha = 1$ (dashed line); plate b) $\alpha = 3/4$ (continuous line), $\alpha = 2$ (dashed line).

Let us now express our operator (6.4) (with symbol $|\kappa|^\alpha$) more properly as the inverse of a suitable integral operator I^α whose symbol is $|\kappa|^{-\alpha}$. This operator can be found in Marcel Riesz' approach to fractional calculus, see *e.g.* Samko, Kilbas & Marichev (1987-1993) and Rubin (1996).

We recall that for any $\alpha > 0\,,\ \alpha \ne 1\,,3\,,5\,,\ldots$ and for a sufficiently well-behaved function $\phi(x)\,,\ x \in \mathbf{R}$, the Riesz integral or Riesz potential I^α and its image in the Fourier domain read

$$I^\alpha\,\phi(x) := \frac{1}{2\,\Gamma(\alpha)\,\cos(\pi\alpha/2)}\int_{-\infty}^{+\infty}|x-\xi|^{\alpha-1}\,\phi(\xi)\,d\xi \div \frac{\hat{\phi}(\kappa)}{|\kappa|^\alpha}\,. \tag{6.14}$$

On the other hand, the *Riesz* potential can be written in terms of two *Weyl* integrals $I^\alpha_\pm$ according to

$$I^{\alpha}\,\phi(x) = \frac{1}{2\,\cos(\pi\alpha/2)}\,\left[I_{+}^{\alpha}\phi(x) + I_{-}^{\alpha}\phi(x)\right]\,, \tag{6.15}$$

where

$$\begin{cases} I_{+}^{\alpha}\,\phi(x) := \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{-\infty}^{x}(x-\xi)^{\alpha-1}\,\phi(\xi)\,d\xi\,, \\ I_{-}^{\alpha}\,\phi(x) := \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{x}^{+\infty}(\xi-x)^{\alpha-1}\,\phi(\xi)\,d\xi\,. \end{cases} \tag{6.16}$$

Then, at least in a formal way, the *space-fractional* derivative (6.2) turns out to be defined as the opposite of the (left) inverse of the Riesz fractional integral, *i.e.*

$$\frac{d^{\alpha}}{d|x|^{\alpha}}\,\phi(x) := -I^{-\alpha}\,\phi(x) = -\frac{1}{2\,\cos(\pi\alpha/2)}\,\left[I_{+}^{-\alpha}\phi(x) + I_{-}^{-\alpha}\phi(x)\right]\,. \tag{6.17}$$

Notice that (6.14) and (6.17) become meaningless when α is an odd integer number. However, for our range of interest $0 < \alpha \le 2$, the particular case $\alpha = 1$ can be singled out since the corresponding *Green* function is already known, see (6.8). Thus, excluding the case $\alpha = 1$, our *space-fractional* diffusion equation (6.1) can be re-written, $x \in \mathbf{R}$, $t \in \mathbf{R}_0^+$, as

$$\frac{\partial u}{\partial t} = -\mathcal{D}\,I^{-\alpha}\,u\,, \quad u = u(x,t;\alpha)\,, \quad 0 < \alpha \le 2\,,\ \alpha \ne 1\,, \tag{6.18}$$

where the operator $I^{-\alpha}$ is defined by (6.16-17).

Here, in order to evaluate the fundamental solution of the *Cauchy* problem, interpreted as a probability density, we propose a numerical approach, original as far as we know, based on a (symmetric) *random walk* model, discrete in space and time, see also Gorenflo & Mainardi (1998). We shall see how things become highly transparent, in that we properly generalize the classical random-walk argument of the standard diffusion equation to our *space-fractional* diffusion equation (6.18). By so doing, we will be able to provide in the near future a numerical simulation of the related (symmetric) stable distributions in a way analogous to the standard one for the Gaussian law.

The *essential idea* is to approximate the inverse operators $I_{\pm}^{-\alpha}$ by the Grünwald-Letnikov scheme, about which the reader can inform himself from the treatises on fractional calculus, see *e.g.* Oldham & Spanier (1974), Samko, Kilbas & Marichev (1987-1993), Miller & Ross (1993), or in the recent review article by Gorenflo (1997). If h denotes a "small" positive step-length, these approximating operators read

$${}_{h}I_{\pm}^{-\alpha}\,\phi(x) := \frac{1}{h^{\alpha}}\,\sum_{k=0}^{\infty}(-1)^{k}\binom{\alpha}{k}\,\phi(x \mp kh)\,. \tag{6.19}$$

Assume, for simplicity, $\mathcal{D} = 1$, and introduce grid points $x_j = j\,h$ with $h > 0$, $j \in \mathbf{Z}$, and time instances $t_n = n\,\tau$ with $\tau > 0$, $n \in \mathbf{N}_0$. Let there be given probabilities $p_{j,k} \geq 0$ of jumping from point x_j at instant t_n to point x_k at instant t_{n+1} and define probabilities $y_j(t_n)$ of the walker being at point x_j at instant t_n. Then, by

$$y_k(t_{n+1}) = \sum_{j=-\infty}^{\infty} p_{j,k}\, y_j(t_n)\,, \qquad \sum_{k=-\infty}^{\infty} p_{j,k} = \sum_{j=-\infty}^{\infty} p_{j,k} = 1\,, \tag{6.20}$$

with $p_{j,k} = p_{k,j}$, a symmetric random walk (more precisely a symmetric random jump) model is described. With the approximation

$$y_j(t_n) \approx \int_{(x_j-h/2)}^{(x_j+h/2)} u(x,t_n)\,dx \approx h\,u(x_j,t_n)\,, \tag{6.21}$$

and introducing the "scaling parameter"

$$\mu = \frac{\tau}{h^\alpha}\,\frac{1}{2\,|\cos(\alpha\pi/2)|}\,, \tag{6.22}$$

we have solved

$$\frac{y_j(t_{n+1}) - y_j(t_n)}{\tau} = -\,{}_hI^{-\alpha}\,y_j(t_n)\,, \tag{6.23}$$

for $y_j(t_{n+1})$. With this we have proved the existence of a consistent (for $h \to 0$) symmetric random walk approximation to (6.18) by taking
i) for $0 < \alpha < 1$, $0 < \mu \leq 1/2$,

$$\begin{cases} {}_hI^{-\alpha}\,y_j(t_n) = \mu\,\dfrac{h^\alpha}{\tau}\,\left[{}_hI_+^{-\alpha}\,y_j(t_n) + {}_hI_-^{-\alpha}\,y_j(t_n)\right]\,, \\ p_{j,j} = 1 - 2\mu\,,\; p_{j,j\pm k} = \mu\,\left|\binom{\alpha}{k}\right|\,, \quad k \geq 1\,; \end{cases} \tag{6.24}$$

ii) for $1 < \alpha \leq 2$, $0 < \mu \leq 1/(2\alpha)$,

$$\begin{cases} {}_hI^{-\alpha}\,y_j(t_n) = \mu\,\dfrac{h^\alpha}{\tau}\,\left[{}_hI_+^{-\alpha}\,y_{j+1}(t_n) + {}_hI_-^{-\alpha}\,y_{j-1}(t_n)\right]\,, \\ p_{j,j} = 1 - 2\mu\,\alpha\,,\; p_{j,j\pm 1} = \mu\,\left[1 + \binom{\alpha}{2}\right]\,, \\ p_{j,j\pm k} = \mu\,\left|\binom{\alpha}{k+1}\right|\,, \quad k \geq 2\,. \end{cases} \tag{6.25}$$

We note that our random walk model is not only symmetric, but also homogeneous, the transition probabilities $p_{j,j\pm k}$ not depending on the index j.

In the special case $\alpha = 2$ we recover from (6.25) the well-known three-point approximation to the heat equation since $p_{j,j\pm k} = 0$ for $k \geq 2$. This means that for common diffusion only jumps of one step to the right or one to the left or jumps of width zero occur, whereas for $0 < \alpha < 2$ $(\alpha \neq 1)$ arbitrary large jumps occur with power-like decaying probability, as it turns out from the asymptotic analysis for the transition probabilities given in (6.24-25). In fact, as $k \to \infty$, one finds

$$p_{j,j+k} \sim \frac{(\tau/h^\alpha)}{\pi} \, \Gamma(\alpha+1) \, \sin\left(\alpha \frac{\pi}{2}\right) \, k^{-(\alpha+1)} \,, \quad 0 < \alpha < 2 \,. \tag{6.26}$$

This result thus provides the discrete counterpart to the asymptotic behaviour of the long power law tails of the symmetric stable distributions, as predicted by (6.13) when $0 < \alpha < 2$.

7. Conclusions

We have treated two generalizations of the standard, one-dimensional, diffusion equation, namely, the *time-fractional* diffusion equation, and the *symmetric space-fractional* diffusion equation, For these equations we have derived the fundamental solutions using Fourier and Laplace transformations, and have demonstrated their connections to *extremal* and *symmetric* stable probability densities, evolving in time or variable in space.

For the *symmetric space-fractional* diffusion equation we have presented a stationary (in time), homogeneous (in space) symmetric random walk model, discrete in space and time, the step-lengths of the spatial grid and the time lapses between transitions properly scaled. In the limit of infinitesimally fine discretization this model (based on the *Grünwald-Letnikov* approximation to fractional derivatives) is consistent with the continuous diffusion process, ı.e. convergent if interpreted as a difference scheme in the sense of numerical analysis. Further generalizations will be considered in another paper (under preparation), in which we shall give a derivation of all stable densities starting from a more general *space-fractional* diffusion equation,

From the mathematical viewpoint the field of such "fractional" generalizations is fascinatinating since here several mathematical disciplines meet and come to a fruitful interplay: e.g. probability theory and stochastic processes, integro-differential equations, transform theory, special functions, numerical analysis. As can be seen from our for some decades now there has been an ever growing interest among physicists and economists in using the concepts of fractional calculus. Among economists we would like to refer the reader to a volume of papers on the topic of "Fractional Differencing and Long Memory Processes", edited by Baillie & King (1996), which appeared as a special issue in the Journal of Econometrics.

Appendix A: The Riemann-Liouville Fractional Calculus

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order. The term *fractional* is a misnomer, but it is retained in accordance with prevailing usage. This appendix is mostly based on the recent review by Gorenflo & Mainardi (1997). For more details on the classical treatment of fractional calculus the reader is referred to Erdélyi (1954), Oldham & Spanier (1974), Samko *et al.* (1987-1993) and Miller & Ross (1993).

According to the Riemann-Liouville approach to fractional calculus, the notion of a fractional integral of order α ($\alpha > 0$) is a natural consequence of the well-known formula (usually attributed to Cauchy), that reduces the calculation of the $n-$fold primitive of a function $f(t)$ to a single integral of convolution type. In our notation the Cauchy formula reads

$$J^n f(t) := f_n(t) = \frac{1}{(n-1)!} \int_0^t (t-\tau)^{n-1} f(\tau)\, d\tau \,, \quad t > 0 \,, \quad n \in \mathbf{N} \,, \quad (A.1)$$

where $\mathbf{N}$ is the set of positive integers. We note that it follows from this definition that $f_n(t)$ vanishes at $t = 0$ along with its derivatives of order $1, 2, \ldots, n-1$. By convention we require that $f(t)$ and henceforth $f_n(t)$ be a *causal* function, *i.e.* identically vanishing for $t < 0$.

In a natural way one is led to extend the above formula from positive integer values of the index to any positive real values by using the Gamma function. Indeed, noting that $(n-1)! = \Gamma(n)$, and introducing the arbitrary *positive* real number α, one defines the *fractional integral of order* $\alpha > 0$:

$$J^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau \,, \quad t > 0 \,, \quad \alpha \in \mathbf{R}^+ \,, \qquad (A.2)$$

where $\mathbf{R}^+$ is the set of positive real numbers. For consistency, we define $J^0 := \mathbf{I}$ (Identity operator), i.e. we mean $J^0 f(t) = f(t)$. Furthermore, by $J^\alpha f(0^+)$ we mean the limit (if it exists) of $J^\alpha f(t)$ for $t \to 0^+$; this limit may be infinite.

We note the *semigroup property* $J^\alpha J^\beta = J^{\alpha+\beta}$, α, $\beta \ge 0$, which implies the *commutative property* $J^\beta J^\alpha = J^\alpha J^\beta$, and the effect of our operators J^α on the power functions

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}\, t^{\gamma+\alpha} \,, \quad \alpha \ge 0 \,, \quad \gamma > -1 \,, \quad t > 0 \,. \qquad (A.3)$$

These properties are of course a natural generalization of those known to apply when the order is a positive integer.

Introducing the Laplace transform by the notation $\mathcal{L}\,\{f(t)\} := \int_0^\infty e^{-st} f(t)\, dt = \widetilde{f}(s)$, $s \in \mathbf{C}$, and using the sign $\div$ to denote a Laplace transform pair, *i.e.* $f(t) \div \widetilde{f}(s)$, we note the following rule for the Laplace transform of the fractional integral,

$$J^\alpha\, f(t) \div \frac{\widetilde{f}(s)}{s^\alpha}\,, \quad \alpha \geq 0\,, \tag{A.4}$$

which is the generalization of the case with an n-fold repeated integral.

After the notion of a fractional integral, that of a fractional derivative of order α ($\alpha > 0$) becomes a natural requirement and one is tempted to substitute α with $-\alpha$ in the above formulas. However, this generalization needs some care in order to guarantee the convergence of the integrals and preserve the well-known properties of the ordinary derivative of integer order.

Denoting the operator of the derivative of order n, by D^n with $n \in \mathbf{N}$, we first note that $D^n\, J^n = \mathbf{I}$, $J^n\, D^n \neq \mathbf{I}$, $\quad n \in \mathbf{N}$, *i.e.* D^n is the left-inverse (and not the right-inverse) of the corresponding integral operator J^n. In fact, we easily recognize from (A.1) that

$$J^n\, D^n\, f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\, \frac{t^k}{k!}\,, \quad t > 0\,. \tag{A.5}$$

As a consequence we expect D^α to be defined as the left-inverse of J^α. For this purpose, introducing the positive integer m such that $m-1 < \alpha \leq m$, one defines the *fractional derivative of order* $\alpha > 0$:

$$D^\alpha\, f(t) := D^m\, J^{m-\alpha}\, f(t) = \frac{d^m}{dt^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}}\, d\tau\right]\,, \tag{A.6}$$

where $m = 1, 2, \ldots$, and $0 \leq m-1 < \alpha \leq m$. Defining for consistency $D^0 = J^0 = \mathbf{I}$, we can easily see that $D^\alpha\, J^\alpha = \mathbf{I}$, $\alpha \geq 0$, and

$$D^\alpha\, t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)}\, t^{\gamma-\alpha}\,, \quad \alpha \geq 0\,, \quad \gamma > -1\,, \quad t > 0\,. \tag{A.7}$$

Of course, these properties are a natural generalization of those known to apply when the order is a positive integer.

Note the remarkable fact that the fractional derivative $D^\alpha\, f$ is not zero for the constant function $f(t) \equiv 1$ if $\alpha \notin \mathbf{N}$. In fact, (A.7) with $\gamma = 0$ tells us that

$$D^\alpha 1 = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,, \quad \alpha \geq 0\,, \quad t > 0\,. \tag{A.8}$$

This, of course, is $\equiv 0$ for $\alpha \in \mathbf{N}$, due to the poles of the gamma function at the points $0, -1, -2, \ldots$. We now observe that an alternative definition of the fractional derivative, originally introduced by Caputo (1967) (1969) in the late sixties and adopted by Caputo and Mainardi (1971) in the framework of the theory of *linear viscoelasticity*, is

$$D_*^\alpha f(t) := J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau\,, \qquad (A.9)$$

where $m = 1, 2, \dots$, and $0 \le m-1 < \alpha \le m$. This definition is of course more restrictive than (A.6), in that it requires the absolute integrability of the derivative of order m. Whenever we use the operator D_*^α we (tacitly) assume that this condition is met. We easily recognize that in general

$$D^\alpha f(t) := D^m J^{m-\alpha} f(t) \neq J^{m-\alpha} D^m f(t) := D_*^\alpha f(t)\,, \qquad (A.10)$$

unless the function $f(t)$ along with its first $m-1$ derivatives vanishes at $t = 0^+$. In fact, assuming that the passage of the m-derivative under the integral is legitimate, one recognizes that, for $m-1 < \alpha < m$ and $t > 0$,

$$D^\alpha f(t) = D_*^\alpha f(t) + \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(0^+)\,, \qquad (A.11)$$

and therefore, recalling the fractional derivative of the power functions (A.7),

$$D^\alpha \left(f(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} f^{(k)}(0^+) \right) = D_*^\alpha f(t)\,. \qquad (A.12)$$

The alternative definition (A.9) for the fractional derivative thus incorporates the initial values of the function and of its integer derivatives of lower order. The subtraction of the Taylor polynomial of degree $m-1$ at $t = 0^+$ from $f(t)$ is a sort of regularization of the fractional derivative. In particular, according to this definition, the relevant property for which the fractional derivative of a constant is still zero can be easily recognized, i.e.

$$D_*^\alpha 1 \equiv 0\,, \quad \alpha > 0\,. \qquad (A.13)$$

We now explore the most relevant differences between the two fractional derivatives (A.6) and (A.9). We agree to denote (A.9) as the *Caputo fractional derivative* to distinguish it from the standard Riemann-Liouville fractional derivative (A.6). We observe, again by looking at (A.7), that $D^\alpha t^{\alpha-1} \equiv 0\,,\ \alpha > 0\,,\ t > 0\,.$

From the above we are thus lead to recognize the following statements about functions which for $t > 0$ admit the same fractional derivative of order α, with $m-1 < \alpha \le m\,,\ m \in \mathbf{N}\,,$

$$D^\alpha f(t) = D^\alpha g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{\alpha-j}\,, \qquad (A.14)$$

$$D_*^\alpha\, f(t) = D_*^\alpha\, g(t) \iff f(t) = g(t) + \sum_{j=1}^{m} c_j\, t^{m-j}\,. \qquad (A.15)$$

In these formulae the coefficients c_j are arbitrary constants.

For the two definitions we also note a difference with respect to the *formal* limit as $\alpha \to (m-1)^+$; from (A.6) and (A.9) we obtain respectively,

$$D^\alpha\, f(t) \to D^m\, J\, f(t) = D^{m-1}\, f(t)\,; \qquad (A.16)$$

$$D_*^\alpha\, f(t) \to J\, D^m\, f(t) = D^{m-1}\, f(t) - f^{(m-1)}(0^+)\,. \qquad (A.17)$$

We now consider the *Laplace transform* of the two fractional derivatives. For the standard fractional derivative D^α the Laplace transform, assumed to exist, requires the knowledge of the (bounded) initial values of the fractional integral $J^{m-\alpha}$ and of its integer derivatives of order $k = 1, 2, \dots, m-1$. The corresponding rule reads, in our notation,

$$D^\alpha\, f(t) \div s^\alpha\, \widetilde{f}(s) - \sum_{k=0}^{m-1} D^k\, J^{(m-\alpha)}\, f(0^+)\, s^{m-1-k}\,, \qquad (A.18)$$

where $m-1 < \alpha \le m$.

The *Caputo fractional derivative* appears to be more suitable for treatment by the Laplace transform technique in that it requires the knowledge of the (bounded) initial values of the function and of its integer derivatives of order $k = 1, 2, \dots, m-1$, in analogy with the case when $\alpha = m$. In fact, by using (A.4) and noting that

$$J^\alpha\, D_*^\alpha\, f(t) = J^\alpha\, J^{m-\alpha}\, D^m\, f(t) = J^m\, D^m\, f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\, \frac{t^k}{k!}\,, \qquad (A.19)$$

we can easily prove the following rule for the Laplace transform,

$$D_*^\alpha\, f(t) \div s^\alpha\, \widetilde{f}(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\, s^{\alpha-1-k}\,, \quad m-1 < \alpha \le m\,. \qquad (A.20)$$

Indeed, the result (A.20), first stated by Caputo (1969) by using the Fubini-Tonelli theorem, appears the most "natural" generalization of the corresponding result well-known for $\alpha = m$.

Gorenflo and Mainardi (1997) have pointed out the utility of the Caputo fractional derivative in the treatment of differential equations of fractional order for *physical applications*. In fact, in physical problems, the initial conditions are usually expressed in terms of a given number of bounded values assumed by the field variable and its derivatives of integer order, despite the fact that the governing evolution equation may be a generic integro-differential equation and therefore, in particular, a fractional differential equation.

We now analyse the simplest differential equations of fractional order, including those which, by means of fractional derivatives, generalize the well-known ordinary differential equations related to relaxation and oscillation phenomena. Generally speaking, we consider the following differential equation of fractional order $\alpha > 0$,

$$D_*^\alpha \, u(t) = D^\alpha \left(u(t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} \, u^{(k)}(0^+) \right) = -u(t) + q(t) \,, \quad t > 0 \,, \quad (A.21)$$

where $u = u(t)$ is the field variable and $q(t)$ is a given function. Here m is a positive integer uniquely defined by $m-1 < \alpha \le m$, which provides the number of the prescribed initial values $u^{(k)}(0^+) = c_k$, $k = 0, 1, 2, \ldots, m-1$. Implicit in the form of (A.21) is our desire to obtain solutions $u(t)$ for which the $u^{(k)}(t)$ are continuous. In particular, the cases of *fractional relaxation* and *fractional oscillation* are obtained for $0 < \alpha < 1$ and $1 < \alpha < 2$, respectively

The application of the Laplace transform through the Caputo formula (A.20) yields

$$\tilde{u}(s) = \sum_{k=0}^{m-1} c_k \, \frac{s^{\alpha-k-1}}{s^\alpha + 1} + \frac{1}{s^\alpha + 1} \, \tilde{q}(s) \,. \quad (A.22)$$

Now, in order to obtain the Laplace inversion of (A.22), we need to recall the *Mittag-Leffler* function of order $\alpha > 0$, $E_\alpha(z)$. This function, named after the great Swedish mathematician who introduced it at the beginning of this century, is defined by the following series and integral representation, valid in the whole complex plane,

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} = \frac{1}{2\pi i} \int_{Ha} \frac{\sigma^{\alpha-1} \, \mathrm{e}^\sigma}{\sigma^\alpha - z} \, d\sigma \,, \quad \alpha > 0 \,. \quad (A.23)$$

Here Ha denotes the Hankel path, i.e. a loop which starts and ends at $-\infty$ and encircles the circular disk $|\sigma| \le |z|^{1/\alpha}$ in the positive sense. It turns out that $E_\alpha(z)$ is an entire function of order $\rho = 1/\alpha$ and type 1.

The *Mittag-Leffler* function provides a simple generalization of the exponential function, to which it reduces for $\alpha = 1$. Particular cases from which elementary functions are easily recognized are

$$E_2\left(+z^2\right) = \cosh\, z \,, \qquad E_2\left(-z^2\right) = \cos\, z \,, \qquad z \in \mathbf{C} \,, \quad (A.24)$$

and

$$E_{1/2}(\pm z^{1/2}) = \mathrm{e}^{z}\,\left[1 + \mathrm{erf}\,(\pm z^{1/2})\right] = \mathrm{e}^{z}\,\mathrm{erfc}\,(\mp z^{1/2})\,, \quad z \in \mathbf{C}\,, \qquad (A.25)$$

where erf (erfc) denotes the (complementary) error function defined as

$$\mathrm{erf}\,(z) := \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-u^2}\,du\,, \quad \mathrm{erfc}\,(z) := 1 - \mathrm{erf}\,(z)\,, \quad z \in \mathbf{C}\,.$$

A noteworthy property of the *Mittag-Leffler* function is based on the following *duplication formula*

$$E_\alpha(z) = \frac{1}{2}\,\left[E_{\alpha/2}(+z^{1/2}) + E_{\alpha/2}(-z^{1/2})\right]\,. \qquad (A.26)$$

In (A.25-26) we agree to denote by $z^{1/2}$ the main branch of the complex root of z.

The *Mittag-Leffler* function is connected to the Laplace integral through the equation

$$\int_0^\infty \mathrm{e}^{-u}\,E_\alpha\,(u^\alpha\,z)\;du = \frac{1}{1-z} \quad \alpha > 0\,. \qquad (A.27)$$

The integral on the L.H.S. was evaluated by Mittag-Leffler who showed that the region of its convergence contains the unit circle and is bounded by the line $\mathrm{Re}\,z^{1/\alpha} = 1$. The above integral is fundamental to the evaluation of the Laplace transform of $E_\alpha\,(-\lambda\,t^\alpha)$ with $\alpha > 0$ and $\lambda \in \mathbf{C}$. In fact, putting in (A.27) $u = st$ and $u^\alpha\,z = -\lambda\,t^\alpha$ with $t \ge 0$ and $\lambda \in \mathbf{C}$, we get the Laplace transform pair

$$E_\alpha\,(-\lambda\,t^\alpha) \div \frac{s^{\alpha-1}}{s^\alpha + \lambda}\,, \quad Re\,s > |\lambda|^{1/\alpha}\,. \qquad (A.28)$$

Then, using (A.28), we put for $k = 0, 1, \ldots, m-1$,

$$u_k(t) := J^k e_\alpha(t) \;\div\; \frac{s^{\alpha-k-1}}{s^\alpha + 1}\,, \quad e_\alpha(t) := E_\alpha(-t^\alpha)\,, \qquad (A.29)$$

and, from inversion of the Laplace transforms in (A.22), we find

$$u(t) = \sum_{k=0}^{m-1} c_k\,u_k(t) - \int_0^t q(t-\tau)\,u_0'(\tau)\,d\tau\,. \qquad (A.30)$$

In particular, formula (A.30) encompasses the solutions for $\alpha = 1\,,\,2$, since $e_1(t) = \exp(-t)$, $e_2(t) = \cos t$. When α is not an integer, namely for $m-1 < \alpha < m$, we note that $m-1$ represents the integer part of α (usually denoted by $[\alpha]$) and m the number of initial conditions necessary and sufficient to ensure the uniqueness of the solution $u(t)$. Thus the m functions $u_k(t) = J^k e_\alpha(t)$ with $k = 0, 1, \ldots, m-1$ represent those particular solutions of the *homogeneous* equation which satisfy the initial conditions $u_k^{(h)}(0^+) = \delta_{k\,h}\,,\ \ h, k = 0, 1, \ldots, m-1\,,$ and therefore they represent the *fundamental solutions* of the fractional equation (A.21), in analogy with the case $\alpha = m$. Furthermore, the function $u_\delta(t) = -u_0'(t) = -e_\alpha'(t)$ represents the *impulse-response solution.*

The *Mittag-Leffler* function of order less than one turns out to be related through the Laplace integral to another special function of Wright type, denoted by $M(z, \nu)$ with $0 < \nu < 1$, following the notation introduced by Mainardi (1994, 1995). Since this function turns out to be relevant in the general framework of fractional calculus with special regard to stable probability distributions, we are going to summarize its basic properties. For more details on this function, see Mainardi (1997), Appendix A.

Let us first recall the more general *Wright* function $W_{\lambda,\mu}(z)\,, z \in \mathbf{C}\,,$ with $\lambda > -1$ and $\mu > 0$. This function, named after the British mathematician who introduced it between 1933 and 1941, is defined by the following series and integral representation, valid in the whole complex plane,

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\lambda n + \mu)} = \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\,\sigma\, +\, z\sigma^{-\lambda}}\, \frac{d\sigma}{\sigma^{\mu}}\,, \qquad (A.31)$$

where Ha denotes the Hankel path. It is possible to prove that the *Wright* function is entire of order $1/(1+\lambda)$, hence of exponential type if $\lambda \geq 0$. The case $\lambda = 0$ is trivial since $W_{0,\mu}(z) = \mathrm{e}^{\,z}/\Gamma(\mu)$. The case $\lambda = -\nu\,,\ \mu = 1-\nu$ with $0 < \nu < 1$ gives the function $M(z, \nu)$, which is of special interest to us. Specifically, we have

$$M(z;\nu) := W_{-\nu,1-\nu}(-z) = \frac{1}{\nu\, z}\, W_{-\nu,0}(-z)\,, \quad 0 < \nu < 1\,, \qquad (A.32)$$

and therefore from (A.31-32)

$$\begin{aligned} M(z;\nu) = \ & \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!}\, \Gamma(\nu\, n)\, \sin\,(\nu\, n\, \pi) = \\ & \frac{1}{2\pi i} \int_{Ha} \mathrm{e}^{\,\sigma\, -\, z\sigma^{\nu}}\, \frac{d\sigma}{\sigma^{1-\nu}}\,, \quad 0 < \nu < 1\,. \end{aligned} \qquad (A.33)$$

In the series representation we have used the reflection formula for the gamma function, $\Gamma(x)\,\Gamma(1-x) = \pi/\sin \pi x$.

Explicit expressions of $M(z;\nu)$ in terms of simpler known functions are expected in particular cases when ν is a rational number. Relevant cases are $\nu = 1/2\,,\ 1/3$ for which

$$M(z;1/2) = \frac{1}{\sqrt{\pi}} \exp\left(-z^2/4\right), \qquad (A.34)$$

$$M(z;1/3) = 3^{2/3}\,\mathrm{Ai}\left(z/3^{1/3}\right), \qquad (A.35)$$

where Ai denotes the *Airy* function.

When the argument is real and positive, *i.e.* $z = r > 0$, the existence of the Laplace transform of $M(r;\nu)$ is ensured by the asymptotic behaviour, as derived by Mainardi & Tomirotti (1995), as $r \to +\infty$,

$$M(r/\nu;\nu) \sim a(\nu)\, r^{(\nu-1/2)/(1-\nu)} \exp\left[-b(\nu)\, r^{1/(1-\nu)}\right], \qquad (A.36)$$

where $a(\nu) = 1/\sqrt{2\pi\,(1-\nu)}$, $b(\nu) = (1-\nu)/\nu$.

It is an instructive exercise to derive the Laplace transform by interchanging the Laplace integral with the Hankel integral in (A.33) and recalling the integral representation (A.23) of the Mittag-Leffler function. We obtain the Laplace transform pair

$$M(r;\nu) \div E_\nu(-s), \quad 0 < \nu < 1. \qquad (A.37)$$

For $\nu = 1/2$, (A.37) with (A.25) and (A.34) provides the result, see *e.g.* Doetsch (1974),

$$M(r;1/2) := \frac{1}{\sqrt{\pi}} \exp\left(-r^2/4\right) \div E_{1/2}(-s) := \exp\left(s^2\right) \mathrm{erfc}\,(s). \qquad (A.38)$$

It should be noted that, since $M(r,\nu)$ is not of exponential order, transforming the Taylor series of $M(r;\nu)$ term by term yields a series of negative powers of s, which represents the asymptotic expansion of $E_\nu(-s)$ as $s \to \infty$ in a certain sector around the real axis.

We also note that (A.37) with (A.23) allows us to compute the moments of any real order $\delta \geq 0$ of $M(r;\nu)$ in the positive real axis. We obtain

$$\int_0^{+\infty} r^{\delta}\, M(r;\nu)\, dr = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)}, \quad \delta \geq 0. \qquad (A.39)$$

When δ is an integer we note that the moments are provided by the derivatives of the Mittag-Leffler function at the origin, i.e.

$$\int_0^{+\infty} r^{n}\, M(r;\nu)\, dr = \lim_{s\to 0} (-1)^n \frac{d^n}{ds^n} E_\nu(-s) = \frac{\Gamma(n+1)}{\Gamma(\nu n+1)}, \qquad (A.40)$$

where $n = 0, 1, 2, \ldots$. The normalization condition $\int_0^\infty M(r;\nu)\, dr = E_\nu(0) = 1$ is recovered for $n = 0$. The relation with the Mittag-Leffler function stated in (A.40) can be extended to the moments of non integer order if we replace the ordinary derivative of order n with the corresponding fractional derivative, of order $\delta \neq n$, in the *Caputo* sense.

Another exercise on the function M concerns the inversion of the Laplace transform $\exp(-s^\nu)$, either by the complex integral formula or by the formal series method. We obtain the Laplace transform pair

$$\frac{\nu}{r^{\nu+1}} M\left(1/r^\nu ; \nu\right) \div \exp\left(-s^\nu\right), \quad 0<\nu<1 . \qquad (A.41)$$

For $\nu = 1/2$, (A.41) with (A.34) provides the known result, see *e.g.* Doetsch (1974),

$$\frac{1}{2\, r^{3/2}} M(1/r^{1/2}; 1/2) := \frac{1}{2\sqrt{\pi}\, r^{3/2}} \exp\left[-1/(4r)\right] \div \exp\left(-s^{1/2}\right) . \quad (A.42)$$

We recall that a rigorous proof of (A.41) was formerly given by Pollard (1946), based on a formal result by Humbert (1945). The Laplace transform pair was also obtained by Mikusiński (1959) and, independently of the earlier results, by Buchen & Mainardi (1975) in a formal way.

Appendix B: The Stable Probability Distributions

Stable distributions are a fascinating and fruitful area of research in probability theory; furthermore, nowadays, they provide valuable models in physics, astronomy, economics, and communication theory.

The general class of stable distributions was introduced and given this name by the French mathematician Paul Lévy in the early 1920's, see Lévy (1924, 1925). The inspiration for Lévy was the desire to generalize the celebrated *Central Limit Theorem*, according to which any probability distribution with finite variance belongs to the domain of attraction of the Gaussian distribution.

Formerly, the topic attracted only moderate attention from the leading experts, though there were enthusiasts, most notably the Russian mathematician Alexander Yakovlevich Khintchine. The concept of stable distributions took full shape in 1937 with the appearance of Lévy's monograph, see Lévy (1937-1954), soon followed by Khintchine's monograph, see Khintchine (1938).

The theory and properties of stable distributions are discussed in some classical books on probability theory including Gnedenko & Kolmogorov (1949-1954), Lukacs (1960-1970), Feller (1966-1971), Breiman (1968-1992), Chung (1968-1974) and Laha & Rohatgi (1979). Also treatises on fractals devote particular attention to stable distributions in view of their properties of scale invariance, see *e.g.* Mandelbrot (1982) and Takayasu (1990).

It is only recently that monographs devoted solely to stable distributions and related stochastic processes have appeared, i.e. Zolotarev (1983-1986), Janiki & Weron (1994), and Samorodnitsky & Taqqu (1994). In this last book tables and graphs related to stable distributions are also exhibited. Previous sets of tables and graphs were provided by Mandelbrot & Zarnfaller (1959), Fama & Roll (1968), Bo'lshev et al. (1968) and Holt & Crow (1973).

Stable distributions have three *exclusive* properties, which can be briefly summarized stating that they 1) are *invariant under addition*, 2) possess their *own domain of attraction*, and 3) admit a *canonical characteristic function*.

Let us now illustrate the above properties which, under necessary and sufficient conditions, can be regarded as equivalent definitions for a stable distribution. We recall the basic results without proof.

A random variable X is said to have a stable distribution $P(x) =$ Prob $\{X \le x\}$ if for any $n \ge 2$, there is a positive number c_n and a real number d_n such that

$$X_1 + X_2 + \ldots + X_n \stackrel{d}{=} c_n\, X + d_n\,, \qquad (B.1)$$

where $X_1, X_2, \ldots X_n$ denote mutually independent random variables with common distribution $P(x)$ with X. Here the notation $\stackrel{d}{=}$ denotes equality in distribution, *i.e.* means that the random variables on both sides have the same probability distribution.

When mutually independent random variables have a common distribution [shared with a given random variable X], we also refer to them as independent, identically distributed (i.i.d) random variables [independent copies of X]. In general, the sum of i.i.d. random variables becomes a random variable with a distribution of different form. However, for independent random variables with a common *stable* distribution, the sum obeys a distribution of the same type, which differs from the original one only in(c_n) and possibly in shift (d_n). When in (B.1) the $d_n = 0$ the distribution is called *strictly stable*.

It is known, see Feller (1966-1971), that the normaliyation constants in (B.1) are of the form

$$c_n = n^{1/\alpha} \quad \text{with} \quad 0 < \alpha \le 2\,. \qquad (B.2)$$

The parameter α is called the *characteristic exponent* or the *index of stability* of the stable distribution.

We agree to use the notation $X \sim P_\alpha(x)$ to denote that the random variable X has a stable probability distribution with characteristic exponent α. We simply refer to $P(x)$, $p(x) := dP/dx$ (probability density function = *pdf*) and X as *α-stable distribution, density and random variable*, respectively.

Definition (B.1) together with theorem (B.2) can be stated in an alternative way requiring only two i.i.d. random variables, see also Lukacs (1960-1970). *A random variable X is said to have a stable distribution if for any positive numbers A and B, there is a positive number C and a real number D such that*

$$A\,X_1 + B\,X_2 \stackrel{d}{=} C\,X + D\,, \tag{B.3}$$

where X_1 and X_2 are independent copies of X. Then there is a number $\alpha \in (0,2]$ such that the number C in (B.3) satisfies $C^\alpha = A^\alpha + B^\alpha$.

For a strictly stable distribution (B.3) holds with $D = 0$. This implies that all linear combinations of i.i.d. random variables obeying a strictly stable distribution are random variables with the same type of distribution.

A stable distribution is called *symmetric* if the random variable $-X$ has the same distribution. Of course, a *symmetric* stable distribution is necessarily *strictly stable*.

Noteworthy examples of stable distributions are provided by the Gaussian (or normal) law (with $\alpha = 2$) and by the Cauchy-Lorentz law ($\alpha = 1$). The corresponding *pdf*′s are

$$p_G(x;\sigma,\mu) := \frac{1}{\sqrt{2\pi}\,\sigma}\,\mathrm{e}^{-(x-\mu)^2/(2\sigma^2)}\,, \quad x \in \mathbf{R}\,, \tag{B.4}$$

where σ^2 denotes the variance and μ the mean, and

$$p_C(x;\gamma,\delta) := \frac{1}{\pi}\,\frac{\gamma}{(x-\delta)^2+\gamma^2}\,, \quad x \in \mathbf{R}\,, \tag{B.5}$$

where γ denotes the semi-interquartile range and δ the "shift".

Another (equivalent) definition states that stable distributions are the only distributions that can be obtained as limits of normalized sums of i.i.d. random variables. A random variable X is stable if it has a *domain of attraction*, if there is a sequence of i.i.d. random variables $Y_1, Y_2, \dots$ and sequences of positive numbers $\{\gamma_n\}$ and real numbers $\{\delta_n\}$, such that

$$\frac{Y_1 + Y_2 + \dots Y_n}{\gamma_n} + \delta_n \stackrel{d}{\Rightarrow} X\,. \tag{B.6}$$

The notation $\stackrel{d}{\Rightarrow}$ denotes convergence in distribution.

It is easy to see that the previous definition (B.1) yields (B.6), e.g., by taking the Y_i's to be independent and distributed like X. The converse is also easy to show, see Gnedenko & Kolmogorov (1949-1954). Therefore we can alternatively state that *a random variable X is stable if it has a domain of attraction.*

When X is Gaussian and the Y_i's are i.i.d. with finite variance, then (B.6) gives the ordinary *central limit theorem.* The domain of attraction of X is said to be *normal* when $\gamma_n = n^{1/\alpha}$; in general, $\gamma_n = n^{1/\alpha}\, h(n)$ where $h(x)\,,\, x > 0\,,$ is a slowly varying function at infinity, that is, $\lim_{x\to\infty} h(ux)/h(x) = 1$ for all $u > 0$, see Feller (1971). The function $h(x) = \log x$, for example, is slowly varying at infinity.

Another definition specifies the *canonical form* that the *characteristic function* (cf) of a stable distribution of index α must have. Recalling that the cf is the Fourier transform of the *pdf*, we use the notation $\hat{p}_\alpha(\kappa) := \langle \exp{(i\kappa X)}\rangle \div p_\alpha(x)$. We first note that a stable distribution is also *infinitely divisible*, *i.e.* for every positive integer n its cf can be expressed as the nth power of some cf. In fact, using the characteristic function, the relation (B.1) is transformed into

$$[\hat{p}_\alpha(\kappa)]^n = \hat{p}_\alpha(c_n\,\kappa)\,\mathrm{e}^{id_n\kappa}\,. \qquad (B.7)$$

The functional equation (B.7) can be solved completely and the solution is known to be

$$\hat{p}_\alpha(\kappa;\beta,\gamma,\delta) = \exp\left\{i\delta\kappa - \gamma^\alpha\,|\kappa|^\alpha\,\left[1 + i\,(\mathrm{sign}\,\kappa)\,\beta\,\omega(|\kappa|,\alpha)\right]\right\}\,, \qquad (B.8)$$

where

$$\omega(|\kappa|,\alpha) = \begin{cases} \tan{(\alpha\,\pi/2)}\,, & \text{if} \quad \alpha \neq 1\,, \\ -(2/\pi)\log|\kappa|\,, & \text{if} \quad \alpha = 1\,. \end{cases} \qquad (B.9)$$

Consequently, *a random variable X is stable if there are four real parameters $\alpha,\beta,\gamma,\delta$ with $0 < \alpha \le 2$, $-1 \le \beta \le +1$, $\gamma > 0$, such that its characteristic function has the canonical form (B.8-9).* We then write $p_\alpha(x;\beta,\gamma,\delta) \div \hat{p}_\alpha(\kappa;\beta,\gamma,\delta)$ and $X \sim P_\alpha(x;\beta,\gamma,\delta)$, partly following the notation of Holt & Crow (1973) and Samorodnitsky & Taqqu (1994).

We note in (B.8-9) that β appears with different signs for $\alpha \neq 1$ and $\alpha = 1$. This minor point has been the source of great confusion in the literature, see Hall (1980) for a discussion. The presence of the logarithm for $\alpha = 1$ is the source of many difficulties, so this case often needs to be treated separately.

The cf (B.8-9) turns out to be a useful tool for studying α-stable distributions and for providing an interpretation of the additional parameters, β (*skewness parameter*), γ (*scale parameter*) and δ (*shift parameter*), see Samorodnitsky & Taqqu (1994). When $\alpha = 2$ the cf refers to the Gaussian distribution with variance $\sigma^2 = 2\,\gamma^2$ and mean $\mu = \delta$; in this case the value of the skewness parameter β is not specified because $\tan\pi = 0$, and one conventionally takes $\beta = 0$.

One easily recognizes that a stable distribution is *symmetric* if and only if $\beta = \delta = 0$ and is symmetric about δ if and only if $\beta = 0$. Stable distributions with extremal values of the skewness parameter are called *extremal*. One can prove that all extremal stable distributions with $0 < \alpha < 1$ are one-sided, the support being $\mathbf{R}_0^+$ if $\beta = -1$, and $\mathbf{R}_0^-$ if $\beta = +1$.

For the stable distributions $P_\alpha(x;\beta,\gamma,\delta)$ we now consider the asymptotic behaviour of the tail probabilities, $T^+(\lambda) := \text{Prob}\{X > \lambda\}$ and $T^-(\lambda) := \text{Prob}\{X < -\lambda\}$, as $\lambda \to \infty$. For the Gaussian case $\alpha = 2$ the result is well known, see e.g. Feller (1957),

$$\alpha = 2 : \quad T^\pm(\lambda) \sim \frac{1}{2\sqrt{\pi}\,\gamma}\,\frac{\mathrm{e}^{-\lambda^2/(4\gamma^2)}}{\lambda}\,, \quad \lambda \to \infty\,. \tag{B.10}$$

Because of the above exponential decay, all the moments of the corresponding *pdf* turn out to be finite, which is an exclusive property of this stable distribution. For all the other stable distributions, the singularity of the characteristic function at the origin is responsible for the algebraic decay of the tail probabilities as indicated below, see e.g. Samorodnitsky & Taqqu (1994),

$$0 < \alpha < 2 : \quad \lim_{\lambda\to\infty} \lambda^\alpha\, T^\pm(\lambda) = C_\alpha\, \gamma^\alpha\, (1 \mp \beta)/2\,, \tag{B.11}$$

where

$$C_\alpha = \left(\int_0^\infty x^{-\alpha}\, \sin x\, dx\right)^{-1} = \begin{cases} \frac{1-\alpha}{\Gamma(2-a)\,\cos(\alpha\pi/2)}\,, & \text{if} \quad \alpha \neq 1\,, \\ 2/\pi\,, & \text{if} \quad \alpha = 1\,. \end{cases} \tag{B.12}$$

We note that for extremal distributions ($\beta = \pm 1$) the above algebraic decay holds true only for one tail, the left one if $\beta = +1$, the right one if $\beta = -1$. The other tail is either identically zero if $0 < \alpha < 1$ (the distribution is one-sided!), or exhibits an exponential decay if $1 \le \alpha < 2$. Because of the algebraic decay, we recognize that

$$0 < \alpha < 2 : \quad \int_{|x|>\lambda} p_\alpha(x;\beta,\gamma,\delta)\, dx = O(\lambda^{-\alpha})\,, \tag{B.13}$$

so the absolute moments of a stable non-Gaussian *pdf* turn out to be finite if their order ν is $0 \le \nu < \alpha$ and infinite if $\nu \ge \alpha$. We are now convinced that the Gaussian distribution is the unique stable distribution with finite variance. Furthermore, when $\alpha \le 1$, the first absolute moment $\langle |X| \rangle$ is infinite as well, so in this case we we need to use the median instead of the expectation value.

There is, however, a fundamental property shared by all stable distributions which we would like to point out: for any α the stable *pdf's* are *unimodal* and indeed *bell-shaped*, i.e. their n-th derivative has exactly n zeros, see Gawronski (1964).

We now come back to the cf of a stable distribution, in order to provide for $\alpha \neq 1$ and $\delta = 0$ a simpler canonical form which allows us to derive convergent and asymptotic power series for the corresponding *pdf*. We first note that the two parameters γ and δ in (B.8), being related to a scale transformation and a translation, are not so essential, since they do not change the shape of the distributions. If we take $\gamma = 1$ and $\delta = 0$, we obtain the so-called *standardized* form of the stable distribution and $X \sim P_\alpha(x;\beta,1,0)$ is referred to as the α-stable *standardized* random variable. Furthermore, we can choose the scale parameter γ in such a way as to get from (B.8-9) the simplified canonical form used by Feller (1952, 1966-1971) and Takayasu (1990) for strictly stable distributions ($\delta = 0$) with $\alpha \neq 1$, which reads in an *ad hoc* notation:

$$\hat{q}_\alpha(\kappa;\theta) := \int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa\, y}\, q_\alpha(y;\theta)\, dy = \exp\left\{-|\kappa|^\alpha\, \mathrm{e}^{\pm i\,\theta\,\pi/2}\right\}, \qquad (B.14)$$

where the symbol $\pm$ takes the sign of κ. This canonical form, which we refer to as the *Feller canonical form*, is derived from (B.8-9) if, in addition to $\alpha \neq 1$ and $\delta = 0$ we require

$$\gamma^\alpha = \cos\left(\theta\,\frac{\pi}{2}\right), \quad \tan\left(\theta\,\frac{\pi}{2}\right) = \beta \tan\left(\alpha\,\frac{\pi}{2}\right). \qquad (B.15)$$

Here θ is the *skewness* parameter instead of β and its domain is restricted in the following region (depending on α)

$$|\theta| \le \begin{cases} \alpha, & \text{if} \quad 0 < \alpha < 1, \\ 2-\alpha, & \text{if} \quad 1 < \alpha < 2. \end{cases} \qquad (B.16)$$

Thus, *when we use the Feller canonical form for strictly stable distributions with index* $\alpha \neq 1$ *and skewness* θ, *we implicitly select the scale parameter* γ ($0 < \gamma \le 1$) which is related to α, β and θ by (B.15). Specifically, the random variable $Y \sim Q_\alpha(y;\theta)$ turns out to be related to the *standardized* random variable $X \sim P_\alpha(x;\beta,1,0)$ by the following relations:

$$Y = X/\gamma, \qquad p_\alpha(x;\beta,1,0) = \gamma\, q_\alpha(y = \gamma x;\theta) \qquad (B.17)$$

with

$$\begin{cases} \gamma &= [\cos(\theta\pi/2)]^{1/\alpha}, \\ \theta &= (2/\pi)\arctan[\beta\tan(\alpha\pi/2)], \\ \beta &= \dfrac{\tan(\theta\pi/2)}{\tan(\alpha\pi/2)}. \end{cases} \qquad (B.18)$$

We recognize that $q_\alpha(y,\theta) = q_\alpha(-y,-\theta)$, so the *symmetric* stable distributions are obtained if and only if $\theta = 0$. We note that for the *symmetric* stable distributions we get the identity between the *standardized* and the *Lévy* canonical forms, since in (B.18) $\beta = \theta = 0$ implies $\gamma = 1$. A particular but noteworthy case is provided by $p_2(x;0,1,0) = q_2(y;0)$, corresponding to the Gaussian distribution with variance $\sigma^2 = 2$.

The *extremal* stable distributions, corresponding to $\beta = \pm 1$, are now obtained for $\theta = \pm\alpha$ if $0 < \alpha < 1$, and for $\theta = \mp(2-\alpha)$ if $1 < \alpha < 2$; For these cases, the scaling parameter turns out to be $\gamma = [\cos(|\alpha|\,\pi/2)]^{1/\alpha}$. It may be an instructive exercise to carry out the inversion of the Fourier transform when $\alpha = 1/2$ and $\theta = -1/2$. In this case we obtain the analytical expression for the corresponding extremal stable *pdf*, known as the (one-sided) *Lévy* density,

$$q_{1/2}(y;-1/2) = \frac{1}{2\sqrt{\pi}}\, y^{-3/2}\, \mathrm{e}^{-1/(4y)}\,, \quad y \ge 0\,. \qquad (B.19)$$

The *standardized* form for this distribution can be easily obtained from (B.19) using (B.17-18) with $\alpha = 1/2$ and $\theta = -1/2$. We get $\gamma = [\cos(-\pi/4)]^2 = 1/2$, $\beta = -1$, so

$$p_{1/2}(x;-1,1,0) = \frac{1}{2}\, q_{1/2}(x/2;-1/2) = \frac{1}{\sqrt{2\pi}}\, x^{-3/2}\, \mathrm{e}^{-1/(2x)}\,, \qquad (B.20)$$

where $x \ge 0$, in agreement with Holt & Crow (1973) [Sec. 2.13, p. 147].

Feller (1952) obtained from (B.14) the following representations by convergent power series for the stable distributions valid for $y > 0$:
i) for $0 < \alpha < 1$ (negative powers)

$$q_\alpha(y;\theta) = \frac{1}{\pi\, y} \sum_{n=1}^{\infty} (-y^{-\alpha})^n\, \frac{\Gamma(n\alpha+1)}{n!}\, \sin\left[\frac{n\pi}{2}(\theta-\alpha)\right], \qquad (B.21)$$

ii) for $1 < \alpha \le 2$ (positive powers)

$$q_\alpha(y;\theta) = \frac{1}{\pi\, y} \sum_{n=1}^{\infty} (-y)^n\, \frac{\Gamma(n/\alpha+1)}{n!}\, \sin\left[\frac{n\pi}{2\alpha}(\theta-\alpha)\right]. \qquad (B.22)$$

The values for $y < 0$ can be obtained from (B.21-22) using the identity $q_\alpha(-y;\theta) = q_\alpha(y;-\theta)$, $y > 0$. As a consequence of the convergence in all of $\mathbf{C}$ of the series in (B.21-22), we recognize that the restrictions of the functions $y\, q_\alpha(y;\theta)$ on the two real semi-axis turn out to be equal to certain *entire* functions of argument $1/|y|^\alpha$ for $0 < \alpha < 1$ and argument $|y|$ for $1 < \alpha \le 2$.

It has been shown, see *e.g.* Bergström (1952), Chao Chung-Jeh (1953), that the two series in (B.21-22) also provide the asymptotic (divergent) expansions to the stable *pdf* with the ranges of α interchanged with those of convergence,

From (B.21-22) a relation between a stable *pdf* with index α and $1/\alpha$ can be derived as noted in Feller (1966-1971). Assuming $1/2 < \alpha < 1$ and $y > 0$, we obtain

$$\frac{1}{y^{\alpha+1}}\, q_{1/\alpha}(y^{-\alpha};\theta) = q_\alpha(y;\theta^*)\,, \quad \theta^* = \alpha(\theta+1)-1\,. \qquad (B.23)$$

A quick check shows that θ^* falls within the prescribed range, $|\theta^*| \le \alpha$, provided that $|\theta| \le 2 - 1/\alpha$.

We now consider two particular cases of the Feller series (B.21-22), of particular interest for us, which turn out to be related to the entire function of the Wright type, $M(z;\nu)$ with $0 < \nu < 1$, reported in Appendix A. These cases correspond to the following extremal distributions

$$\Phi_1(y) := q_\alpha(y;-\alpha)\,, \quad y > 0\,, \quad 0 < \alpha < 1\,, \qquad (B.24)$$

$$\Phi_2(y) := q_\alpha(y;\alpha-2)\,, \quad y > 0\,, \quad 1 < \alpha \le 2\,, \qquad (B.25)$$

for which the Feller series (B.21-22) reduce to

$$\Phi_1(y) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\, y^{-\alpha n-1}\,\frac{\Gamma(n\alpha+1)}{n!}\,\sin\left(n\pi\alpha\right)\,, \quad y > 0\,, \qquad (B.26)$$

and

$$\Phi_2(y) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\, y^{n-1}\,\frac{\Gamma(n/\alpha+1)}{n!}\,\sin\left(\frac{n\pi}{\alpha}\right)\,, \quad y > 0\,. \qquad (B.27)$$

In fact, recalling the series representation of the general Wright function, $W_{\lambda,\mu}(z)$ with $\lambda > -1$, $\mu > 0$, see (A.31), and the definition of the function $M(z;\nu)$ with $0 < \nu < 1$, see (A.32-33), we recognize that

$$\Phi_1(y) = \frac{1}{y} W_{-\alpha,0}(-y^{-\alpha}) = \frac{\alpha}{y^{\alpha+1}}\, M(y^{-\alpha};\alpha)\,, \quad y > 0\,, \qquad (B.28)$$

and

$$\Phi_2(y) = \frac{1}{y}\, W_{-1/\alpha,0}(-y) = \frac{1}{\alpha}\, M(y;1/\alpha)\,, \quad y > 0\,. \qquad (B.29)$$

We would like to remark that the above relations with the Wright functions have also been noted by Engler (1997).

It is worth pointing out that whereas $\Phi_1(y)$ totally represents the one-sided stable *pdf* $q_\alpha(y;-\alpha)$, $0<\alpha<1$, with support in $\mathbf{R}_0^+$, $\Phi_2(y)$ is the restriction on the positive axis of $q_\alpha(y;\alpha-2)$, $1<\alpha\le 2$, whose support is all of $\mathbf{R}$. Since the function $M(z;\nu)$ turns out to be normalized in $\mathbf{R}_0^+$, see (A.39-40), we also note

$$\int_0^\infty \Phi_1(y)\,dy = 1\,; \qquad \int_0^\infty \Phi_2(y)\,dy = 1/\alpha\,. \qquad (B.30)$$

Using the results (A.41) and (A.37) we can easily evaluate the Laplace transforms of $\Phi_1(y)$ and $\Phi_2(y)$, respectively. We obtain

$$\mathcal{L}[\Phi_1(y)] = \widetilde{\Phi}_1(s) = \exp\left(-s^\alpha\right), \quad 0<\alpha<1\,, \qquad (B.31)$$

$$\mathcal{L}[\Phi_2(y)] = \widetilde{\Phi}_2(s) = \frac{1}{\alpha}\,E_{1/\alpha}\,(-s)\,, \quad 1<\alpha\le 2\,, \qquad (B.32)$$

where $E_{1/\alpha}(\cdot)$ denotes the *Mittag-Leffler* function of order $1/\alpha$, see (A.23).

It is an instructive exercise to derive the asymptotic behaviours of $\Phi_1(y)$ and $\Phi_2(y)$ as $y\to 0^+$ and $y\to +\infty$. By using the expressions $(B.28-29)$ in terms of the function M and recalling the series and asymptotic representations of this function, see (A.33) and (A.36), we obtain

$$\Phi_1(y) = \begin{cases} O\left(y^{-(2-\alpha)/[2(1-\alpha)]}\,\mathrm{e}^{-c_1\,y^{-\alpha/(1-\alpha)}}\right), & \text{as} \quad y\to 0^+\,, \\ \dfrac{\alpha}{\Gamma(1-\alpha)}\,y^{-\alpha-1}\,\left[1+O\,\left(y^{-\alpha}\right)\right], & \text{as} \quad y\to +\infty\,, \end{cases} \qquad (B.33)$$

$$\Phi_2(y) = \begin{cases} \dfrac{1/\alpha}{\Gamma(1-1/\alpha)}\,\left[1+O\,\left(y\right)\right], & \text{as} \quad y\to 0^+\,, \\ O\left(y^{(2-\alpha)/[2(\alpha-1)]}\,\mathrm{e}^{-c_2\,y^{\alpha/(\alpha-1)}}\right), & \text{as} \quad y\to +\infty\,, \end{cases} \qquad (B.34)$$

where c_1 , c_2 are positive constants depending on α . We note that the exponential decay is found for $\Phi_1(y)$ as $y\to 0^+$ but as $y\to +\infty$ for $\Phi_2(y)$.

Explicit expressions for *stable pdf*'s can be derived form those for the function $M(z;\nu)$ when $\nu=1/2$ and $\nu=1/3$, given in Appendix A, see (A.34-35). Of course the $\nu=1/2$ expression can be used to recover the well-known (symmetric) Gaussian distribution $q_2(y;0)$ accounting for (B.29), and the (one-sided) Lévy distribution $q_{1/2}(y;-1/2)$, see (B.19), accounting for (B.28). The $\nu=1/3$ expression provides, accounting for (B.28),

$$\begin{aligned} q_{1/3}(y;-1/3) = \;& 3^{-1/3}\,y^{-4/3}\,\mathrm{Ai}\,\left[(3y)^{-1/3}\right] = \\ & \frac{1}{3\pi}\,y^{-3/2}\,K_{1/3}\,\left(2/\sqrt{27y}\right)\,, \end{aligned} \qquad (B.35)$$

where Ai denotes the Airy function and $K_{1/3}$ the modified Bessel function of the second kind of order $1/3$.

The equivalence between the two expressions in (B.35) can be proved in view of the relation :

$$\mathrm{Ai}\,(z) = \frac{1}{\pi}\sqrt{\frac{z}{3}}\,K_{1/3}\left(\frac{2}{3}\,z^{2/3}\right)\,, \qquad (B.36)$$

see Abramowitz & Stegun (1965) [(10.4.14)]. The case $\alpha = 1/3$ has also been discussed by Zolotarev (1983-1986), who quoted the corresponding expression of the *pdf* in terms of $K_{1/3}$.

A general representation of all stable distributions (including the *extremal* distributions considered above) in terms of special functions, has been only recently achieved by Schneider (1986). In his remarkable (but almost entirely neglected) article, Schneider has established that all stable distributions can be characterized in terms of a general class of special functions, called H functions. Note that although these functions usually bear the name of their rediscoverer Charles Fox (1961), they were first introduced by the Italian mathematician Salvatore Pincherle (1888). For details on H functions, see e.g. Mathai & Saxena (1978) and Srivastava et al. (1982).

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