GAUGE PHYSICS OF FINANCE

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1. Informal Sketch

There is a common belief that if nature (and human society, in particular) could be described in some unified terms, fundemental concepts from physics may find their use there. The present paper follows this philosophy and tries to draw parallels between the theory of financial markets and quantum gauge theory. Since it is difficult to be equally comprehensible for both physicists and economists, we first give an introduction, which is basically intuitive and designed to explain the main logic of our considerations. Further sections are more mathematical and use notions of quantum field theory.

1.1. NET PRESENT VALUE AS A PARALLEL TRANSPORT

First of all, let us recall what the NPV method is. The NPV investment method works on the simple but fundamental principle that money has a time value. This time value has to be taken into account through the so-called discounting process.

To elucidate this idea we use an easy example from [2]: "Suppose you were made an offer: if you pay 500 pounds now, you will immediately receive 200 French Francs, 200 Japanese Yen and 200 German Marks. How would you go about deciding whether the offer was worthwhile? What you certainly would not do is to say: I have to give up 500 pieces of paper and, in return, I will get 600 pieces of paper. As I end up with 100 more pieces of paper, the deal is worthwhile! Instead, you would recognize that in order to evaluate the offer, you have to convert all the different currencies to a common currency, and then undertake the comparison" (please, do not consider this as Euro propaganda!). In the same way that money in different currencies cannot be compared directly, but first has to be converted to a common currency, money in the same currency but at different points in time cannot be compared directly, but must first be converted to a common point in time. This reflects the time value of money.

Intuitively it is clear that given the choice between \$100 now or \$100 in one year's time, most people would take the \$100 now, since the money could be put in deposit (risk free investment) at some interest rate, r. Then, in one year's time the \$100 will have turned itself into $\$(1+r)100$ instead of remaining \$100 only. Therefore, r — the interest rate — represents the time value of money.

Skipping here some details (difference between simple and compound interest rate, continuous compounding, flat and effective interest rates [3]), we are now ready to formulate what the NPV is: if an amount of money F is to be received in T years' time, the Present Value of that amount ($NPV(F)$) is the sum of money P (principal) which, if invested today, would generate the compound amount F in T years' time:

$$NPV(F) \equiv P = \frac{F}{(1+r)^T} \ .$$

The interest rate involved in this calculation is known as the discount rate and the term $(1+r)^{-T}$ is known as the T-year discount factor D_T:

$$D_T = (1+r)^{-T} \ . \tag{1}$$

In a similar way, to calculate the present value of a stream of payments, the above formula is applied to each individual payment and the resulting individual present values are then summed. So, the NPV method states that if the NPV of an investment project has zero or positive value, the company should invest in the project; if it has negative value, it should not invest. The NPV is also useful for the comparative analysis of several projects but we do not stop here for the details (see [2]).

What is really important here for our goals is the following geometrical interpretation: the discounting procedure plays the role of a "parallel transport" of an amount of money through time (though in fixed currency). The discounting factor (1) is then an element of a structural group of a fibre bundle (which still has to be defined), and the discount rate coincides with the time component of the connection vector field. Recall that in differential geometry the connection field is responsible for pulling a fibre element from one point of a base space to another. Moreover, it is obvious that the "space" components of the connection have something to do with the exchange rates (see the example above). Indeed, the exchange rates or prices are responsible for converting money in different currencies or different securities (read points of discrete "space") to the same currency (point of the space) at a fixed moment of time. Thus, they can be interpreted as elements of the structural group which "transports" the money in "space" directions and are space analogues of the discount factor. Summing up, we see that there is some analogy between elements of a fibre bundle picture with some connection field and a capital market. We make the analogy precise below.

1.2. ARBITRAGE AS A CURVATURE OF CONNECTION

The next keyword is arbitrage. What is arbitrage then? Basically it means "getting something from nothing" and a free lunch after all. A stricter definition states that arbitrage is an operational opportunity to make a risk-free profit [4] with a rate of return higher than the risk-free interest rate accrued on deposit (a formalized version can be found in [5]).

The arbitrage appears in the theory when we consider the curvature of the connection. In more detail, the rate of excess return for an elementary arbitrage operation (the difference between the rate of return for the operation and the risk-free interest rate) is an element of the curvature tensor calculated from the connection. This can be understood keeping in mind that the elements of the curvature tensor are related to the difference between two results of infinitesimal parallel transports performed in different order. In financial terms this means that the curvature tensor elements measure a difference in gains accrued from two financial operations with the same initial and final points or, in other words, a gain from an arbitrage operation.

In a certain sense, the rate of excess return for an elementary arbitrage operation is the analog of the electromagnetic field. In the absence of any uncertainty (or, in other words, in the absence of "walking" prices, exchange and interest rates) the only state realised is the state of zero arbitrage. However, if we introduce uncertainty in the game, prices and rates move and some virtual arbitrage possibilities appear. Therefore, we can say that uncertainty plays the same role in the theory to be developed here as the quantization did for quantum gauge theory.

Money flows as matter fields

The final ingredients to be added to our theory are "matter" fields that interact through the connection. By now it should be clear that the "matter" fields are money flow fields, which have to be gauged by the connection. Indeed, we started the introduction of the concept with the example of the NPV method, which shows how an amount of money units changes while the payment is "pulled" in time or one currency is converted to another.

Dilatations of money units (which do not change real wealth) play the role of gauge transformations, which eliminate the effect of the dilatation by appropriately tuning the connection (interest rate, exchange rates, prices and so on), exactly as the Fisher formula does for the real interest rate in the case of inflation [2]. *The symmetry of real wealth with respect to a local dilatation of money units (security splits and the like) is the gauge symmetry of the theory.*

Following a formal analogy with the $U(1)$ gauge theory (electrodynamics) case, we can say that the amount of a certain currency at a particular moment in time is analogous to a value of a phase cross-section of the $U(1) \times M$ fibre bundle at some space-time point. If we want to compare values of the $U(1) \times M$ cross-section at different points of space-time, we have to parallel transport the phase from one point of the base to another, exactly as we have to convert one currency to another or discount money in the financial setting.

A theory may contain several types of "matter" fields, which may differ, for example, by a sign of the connection term as is the case for positive and negative charges in electrodynamics. In the financial setting this corresponds to the different preferences of investors. Thus in this paper we will deal with cash flows and debt flows, which behave differently under the same gauge field: cash tries to maximize itself, while debts try to minimize themselves. This is equivalent to the behaviour of positive and negative charges in the same electric field, where they move in different directions.

The strategy of investors is not always optimal. This is due to partially incomplete information, choice procedure, to some extent investor (or manager) internal objectives. This means that the money flows are not certain and fluctuate in the same manner as the prices and rates do. Therefore, this requires a statistical description of the money flows, which, once again, lead us back to an effective quantization of the theory.

1.3. THE NET RESULT: A QED-LIKE SYSTEM

Collecting all this together, we are ready to map the capital market to a system of particles with positive (securities) and negative ("debts") charges, which interact with each other through the electromagnetic field (gauge field of the arbitrage). When there are virtual arbitrage opportunities present, money flows into those regions of configuration space where these appear, while debts try to escape from there. In-flowing positive charges and out-flowing negative ones screen the profitable fluctuation and restore equilibrium, thereby cancelling the arbitrage opportunity.

Taking into account the uncertainty mentioned above, we come to a quantum field theory with a gauge field and "matter" fields of opposite charges. At this point, the standard machinery of quantum field theory may be applied to obtain distribution functions of the interest/exchange rates and cash-debt flow correlators (which are essential for the response of the system). We may use this model to predict the dynamical response of a financial market, establish a dynamical portfolio theory and to approach option pricing and other problems.

The last point to note in this subsection is the principal lattice nature of the capital market theory, since there always exists a natural minimal time interval (transaction time), and so our "space" is a graph. This removes the usual problems of quantum field theory with divergences due to the continuous character of its space-time base.

To make the analogy between Quantum Electrodynamics and Capital Market Theory more transparent, we give a "dictionary" for translating one language into the other:

1.4. BASIC ASSUMPTIONS

In the previous subsections we discussed the main assumption (or postulate), which is the existence of a *local gauge invariance* with the dilatation gauge group. It can be shown that this postulate essentially dictates the dynamical rules of the theory. However, there are a number of other assumptions which we want to list here. For convenience we divide these assumptions into two sets. The first set deals with the prices and rates, i.e. contains assumptions concerning the connection field.

1. Exchange rates, prices of various securities and interest rates fluctuate, providing local arbitrage opportunities.
2. The most probable configuration of the random connection is the configuration with a minimal absolute value of the excess rate of return.
3. Arbitrage opportunities, in the absence of money flows, are generally correlated at different space-time points, which is particularly important for portfolio analysis. However, for the sake of simplicity, we consider here statistically independent arbitrage events.
4. We assume an exponential distribution for the rate of return on a local arbitrage operation. Its characteristics, in general, depend on both

QED	**Capital Market Theory**
Time component of connection	Interest rate
Operator of parallel transport in time	Discount factor
Operators of parallel transport in space	Exchange rates and prices
Electromagnetic field	Excess return on elementary arbitrage operation
Quantization of electromagnetic field	Uncertainty in price movements
Matter fields	Money flow fields
Quantization of matter field	Limited rationality of traders
Gauge transformations	Change of asset units
Gauge invariance	Independence on this change
Positive (negative) charges	Securities in long (short) position

"space" and time coordinates. However, for the sake of simplicity, we omit these details in the discussion below.

5. When a continuous limit will be taken, we assume all required smoothness properties of relevant objects to hold.

The second set of main assumptions concern the behavior of cash-debt flows i.e. the "matter" fields.

1. We assume a perfect capital market environment, i.e. it is always possibile to deposit money and to borrow without any restrictions and at the same interest rate [6].
2. There are transaction costs. Their presence is not just an unimportant complication. Transaction costs play the role of inertia for cash-debt flows and stabilize the system.
3. Investors are (limitedly) rational, and are trying to maximize their gain from the securities, minimizing debts at the same time. The time scale we are thinking about is of the order of several transaction times. On this time scale, a trader does not ponder random processes and portfolio analysis, but concentrates on maximizing his gain on the basis of intuition or technical forecasts.

4. Exactly as the rates and prices may fluctuate and are uncertain, flow trajectories fluctuate around the most profitable trajectory ("classical trajectory"). This reflects the fact that investors do not always behave optimally.

Below we repeat the assumptions in a stricter form, which is useful for the further formalization of the theory.

2. Formal Constructions

In this section we formalize the previous considerations. More precisely, we give a description of the relevant fibre bundles, we construct the parallel transport rules using elements of the structural group and give an interpretation to the parallel transport operators. The corresponding curvature is also defined and is shown to be equal to the rate of excess return on the elementary plaquette arbitrage operation. This opens a way to the construction of the dynamics of the parallel transport factors giving a lattice gauge theory formulation. The construction of the dynamics is then repeated for the case of "matter" fields representing cash-debts flows.

2.1. FIBER BUNDLE CONSTRUCTION

It is well-known that many important concepts in physics can be interpreted in terms of the geometry of fibre bundles [7]. Maxwell's theory of electromagnetism and Yang-Mills theories are essentially theories of the connections on principal bundles with a given gauge group G as the fibre. Einstein's theory of gravitation deals with the Levi-Civita connection on the frame bundle of the spacetime manifold. In this section we show how the construction of fibre bundles can also be applied to describe a framework in which to develop a capital market theory.

Construction of the base

We now have to construct a base for the fibre bundle. Let us order the complete set of assets (which we want to analyze) and label them from 0 to N. This set can be represented by N (asset) points on a 2-dimensional plane (the dimension is a matter of convenience and can be chosen arbitrarily). To add time to the construction, we attach a copy of a Z-lattice (i.e. set of all integer number $\{..., -1, 0, 1, 2, ...\}$) to each asset point. We use discretized time, since there is a natural time step and all real trading happens discretely, anyway. All together this gives the prebase set $L_0 = \{1, 2, ..., N\} \times Z$.

The next step in the construction is to define the *connectivity* of the prebase. To this end, we start with the introduction of a matrix of links $\Gamma : L_0 \times L_0 \rightarrow \{0, \pm 1\}$, which is defined by the following rule for any $x \equiv (i, n) \in L_0$ and $y \equiv (k, m) \in L_0$: $\Gamma(x, y) = 0$ except for

1. $i = k$ and $n = m - 1$ assuming that the i-th security exists at the n-th moment and this moment is not an expiration date for the security;
2. $n = m - 1$ and at n-th moment of time the i-th asset can be exchanged to some quantity of the k-th asset at some rate (we assume that the transaction takes one unit of time).

In the latter situation $\Gamma(x, y) = 1 = -\Gamma(y, x)$.

Using the matrix $\Gamma(.,.)$ we define a *curve* $\gamma(x, y)$ in L_0 which links two points $x, y \in L_0$. We call the set $\gamma(x, y) \equiv \{x_j\}_{j=1}^{j=p}$ a curve in L_0 with ends at points $x, y \in L_0$ and $p - 1$ segments if $x = x_1$, $x_p = y$, $\forall x_j \in L_0$ and

$$\Gamma(x_j, x_{j+1}) = \pm 1 \qquad \text{for} \qquad \forall j = 1, ..., p - 1 \ .$$

The whole L_0 can be divided into a set of connected components. A connected component is a maximal set of elements of L_0, which can be linked by some curve for any pair of elements. The base L is defined now as the connected component containing US dollars at, say, 15.30 on the 17th of June, 1997. This completes the construction of the base of the fibre bundle.

Structural group

The structural group G to be used below is a group of dilatations. The corresponding irreducible representation is the following: the group G is a group of maps g of $R_+ \equiv]0, +\infty)$ to R_+, which acts as a multiplication of any $x \in R_+$ by some positive constant $\lambda(g) \in R_+$:

$$g(x) = \lambda(g) \cdot x \ .$$

Transition functions of a fibre bundle with the structure group correspond below to various swap rates, exchange rates, discount factors for assets.

2.1.1. *Fibres*

In the paper we use fibre bundles with the following fibres F:

1. $F = G$, i.e. the fibre coincides with the structure group. The corresponding fibre bundle is called the principal fibre bundle E_P. A gauge theory in the fibre bundle in the next section corresponds to random walks of prices and rates.
2. $F = R_+$. This fibre bundle will be important for describing cash-debt flows. Indeed, a *cross-section* (or simply a *section*) s (a rule which assigns a preferred point $s(x)$ on each fibre to each point $x \in L$ of the base) of the fibre bundle is a "matter" field. In this context $s(x \equiv (i, m))$ gives the number of units of the i-th asset at the moment of time m.

Actions and the corresponding functional integrals will be written in terms of cross sections of fibre bundles. The main property of the objects (actions and measures of the integrations) will be local gauge invariance, i.e. independence with respect to a local action of the structural group.

The fibre bundle E we use below is trivial, i.e. $E = L \times F$, and we do not digress to define projections. The construction of the fibre bundles for the simple stock exchange, the FX-market and financial derivatives can be found in [1].

2.2. PARALLEL TRANSPORT, CURVATURE AND ARBITRAGE

A connection is a rule for parallel transporting an element of a fibre from one point (x) of a base to another point (y). This means that an operator of the parallel transport along the curve γ, $U(\gamma) : F_x \to F_y$ is an element of the structural group of the fibre bundle [7]. Since we are not dealing with the continuous case but restrict ourselves to a lattice formulation, we do not need to introduce a vector-field of the connection, but rather use elements of the structural group G. By definition, an operator of the parallel transport along a curve γ, $U(\gamma)$, is defined as the product of operators of parallel transport along the links which constitute the curve γ:

$$U(\gamma) = \prod_{i=1}^{p-1} U(x_i, x_{i+1}) \ , \qquad \gamma \equiv \{x_i\}_{i=p-1}^{i=1} \ , \qquad x_1 = x, \quad x_p = y \ .$$

This means that we only need to define the parallel transport operators along elementary links. Since $U(\gamma) = U^{-1}(\gamma^{-1})$, this restricts us to a definition of those along an elementary link with positive connectivity. Summing up, the rules of parallel transport in the fibre bundles are completely defined by a set of parallel transport operators along elementary links with positive connectivity. The definition of the set is equivalent to a definition of the parallel transport in the fibre bundle.

Since in subsection 2.1 the connectivity was defined by a possibility of asset movements in "space" and time, it allows us to give an interpretation to parallel transport. In this subsection two principle kinds of links with positive connectivity were defined. First one connects two points (i, n) and $(i, n+1)$ and represents a deposition of the i-th asset for one unit of time. This deposition then results in a multiplication of the number of asset units by an interest factor (or internal rate of return factor) calculated as:

$$U((i, n), (i, n+1)) = e^{r_i \Delta} \in G \ ,$$

where Δ is a time unit and r_i is an appropriate rate of return for the i-th asset. In the continuous limit r_i becomes a time component of the corresponding connection vector field at the point $(i, \Delta n)$.

In the same way the parallel transport operator is defined for the second kind of the elementary link, i.e. links between (i,n) and $(k,n+1)$ if there is a possibility to change at the n-th moment a unit of the i-th asset to $S_n^{i,k}$ units of the k-th asset:

$$U((i,n),(k,n+1)) = S_n^{i,k} \in G \ .$$

In general, an operator of parallel transport along a curve is a multiplier by which a number of asset units is multiplied as a result of an operation represented by the curve.

Results of parallel transports along two different curves with the same boundary points are not equal for a generic set of parallel transport operators. A measure of the difference is the curvature tensor F. Its elements give the change in multiplier if we parallel transport an asset around an infitesimal elementary plaquette, whose sides are elements from the base L:

$$F_{\text{plaquette}\to 0} = \prod_m U_m - 1 \ .$$

The index m runs over all plaquette links, $\{U_m\}$ are corresponding parallel transport operators, with some convention for the orientation.

Now we show that the elements of the curvature tensor are, in fact, the excess returns on the operation corresponding to the plaquette. Since elements of the curvature tensor are local quantities, it is sufficient to consider an elementary plaquette on a "space"-time base graph. Let us, for example, consider two different assets (for the moment, we will call them share and cash), which can be exchanged to each other with some exchange rate S_i (one share is exchanged for S_i units of cash) at some moment T_i, and the reverse rate (cash to share) is S_i^{-1}. We suppose that there exists a transaction time Δ and this Δ is taken as a time unit. So the exchange rates S_i are quoted at a set of equidistant times: $\{T_i\}_{i=1}^N, T_{i+1} - T_i = \Delta$. The interest rate for cash is r_1, so that between two subsequent times T_i and T_{i+1}, the volume of cash is increased by a factor $e^{r_1\Delta}$. The shares are characterized by a rate r_2. As we will show later, due to gauge invariance, we can fix r_1 to be the risk-free interest rate and r_2 related to the average rate of return of the share.

Let us consider an elementary (arbitrage) operation between two subsequent times T_i and T_{i+2}. There are two possibilities for an investor who possesses a cash unit at the moment T_i, to obtain shares by the moment T_{i+2}. The first one is to put cash on a bank deposit with the interest rate r_1 at T_i, withdraw money at T_{i+2}, and buy shares for at the price of S_{i+1} each. In this way, the investor gets $e^{r_1\Delta}S_{i+1}^{-1}$ shares at the moment T_{i+2} for each unit of cash he had at T_i. The second way is to buy the shares for S_i each at the moment T_i. Then, at T_{i+2} our investor will have $S_i^{-1}e^{r_2\Delta}$ shares for each unit of cash at T_i. If these two numbers ($e^{r_1\Delta}S_{i+1}^{-1}$ and $S_i^{-1}e^{r_2\Delta}$) are not equal, then there is a possibility for arbitrage. Indeed, suppose that $e^{r_1\Delta}S_{i+1}^{-1} < S_i^{-1}e^{r_2\Delta}$, then at the moment T_i an arbitrager can borrow one unit of cash, buy S_i^{-1} shares and get $S_i^{-1}e^{r_2\Delta}S_{i+1}$ units of cash from selling shares at the moment T_{i+1}. The value of this cash discounted to the moment T_i is $S_i^{-1}e^{r_2\Delta}S_{i+1}e^{-r_1\Delta} > 1$. This means that $S_i^{-1}e^{r_2\Delta}S_{i+1}e^{-r_1\Delta} - 1$ is an arbitrage excess return on the operation. On the other hand, as we have shown above, this represent the lattice regularisation of an element of the curvature tensor along the plaquette. If $e^{r_1\Delta}S_{i+1}^{-1} > S_i^{-1}e^{r_2\Delta}$, then an arbitrager can borrow one share at the moment T_i, sell it for S_i units of cash, put cash in the bank and buy $S_ie^{r\Delta}S_{i+1}^{-1}$ shares at T_{i+2}. We have an arbitrage situation again.

We consider the following quantity

$$(S_i^{-1}e^{r_2\Delta}S_{i+1}e^{-r_1\Delta} + S_ie^{r_1\Delta}S_{i+1}^{-1}e^{-r_2\Delta} - 2)/2\Delta \ , \qquad (2)$$

which is the sum of excess returns on the plaquette arbitrage operations. In the continuous limit this quantity converges, as usual, to the square of the curvature tensor element. The absence of arbitrage is equivalent to the equality

$$S_i^{-1}e^{r_2\Delta}S_{i+1}e^{-r_1\Delta} = S_ie^{r_1\Delta}S_{i+1}^{-1}e^{-r_2\Delta} = 1 \ ,$$

and we can use quantity (2) to measure the arbitrage (excess rate of return). More formally, the expression (2) may be written as

$$R = (U_1U_2U_3^{-1}U_4^{-1} + U_3U_4U_2^{-1}U_1^{-1} - 2)/2\Delta \ .$$

In this form it can be generalized to other plaquettes such as, for example, "space"-"space" plaquettes.

As we have seen above, excess returns are elements of the lattice curvature tensor calculated from the connection. In this sense, the rate of excess return for an elementary arbitrage operation is the analog of the electromagnetic field. In the absence of uncertainty (or, in other words, in the absence of walking prices, exchange and interest rates) the only state that is realized is the state of zero arbitrage. However, if we introduce uncertainty into the game, prices and the rates will move, and some virtual arbitrage possibilities appear. Therefore, we can say that uncertainty plays the same role in our theory, as quantization does for Quantum Gauge Theory.

The last point to add in this section is the notion of gauge transformation. Gauge transformation means a local change of a scale in fibres:

$$f_x \to g(x)f_x \equiv f'_x \ , \quad f_x \in F_x \ , \quad g(x) \in G \ , \quad x \in E$$

together with the following transformation of the parallel transport operators:

$$U(y,x) \to g(y)U(y,x)g^{-1}(x) \equiv U'(y,x) \in G \ .$$

It is easy to see that the parallel transport operation commutes with a gauge transformation:

$$g(y)(U(y,x)f_x) = U'(y,x)f'_x \tag{3}$$

and the curvature tensor is invariant under the transformation:

$$U_1U_2U_3^{-1}U_4^{-1} = U'_1U'_2(U'_3)^{-1}(U'_4)^{-1} \ . \tag{4}$$

2.3. GAUGE FIELD DYNAMICS

In the previous subsections we have shown that the exchange rates and the interest rate (or, more generally, the internal rate of return) discount factors are elements of the structural group of the fibre bundle. Moreover, they are responsible for parallel transport in "space" and time directions correspondingly. In the present subsection we address the question of the dynamics for the exchange/discount factors.

At first sight the dynamics is difficult to specify since it is not restricted, and any attempt to formulate the dynamics seems to be arbitrary and not sufficiently motivated. However, as we show below, the dynamics can be derived from a few general and natural ssumptions. The main postulate of the present analysis is an assumption about the local gauge invariance with the dilatation group as the gauge group.

Postulate 1: Gauge invariant dynamics

We assume that *no observable properties of the financial environment (in particular, rules of dynamical processes) depend on the choice of units of the assets*. This means that all effects of, say, change of currency units or share splittings may be eliminated by a corresponding change of interest rates, exchange rates and prices [8]. This is a very natural assumption which allows us, however, to make a step towards the specification of the dynamics. Indeed, due to gauge invariance, the action which governs the dynamics has to be constructed from gauge invariant quantities.

Postulate 2: Locality

Furthermore, we assume *local dynamics for the exchange/interest rate factors*. This locality means that the dynamics of an asset is influenced by connected (in the sense of Γ connectivity on the base graph L) assets only.

These two postulates allow us to make the following conclusion: the action s_{gauge} has to be a sum over plaquettes in the base graph of some function of the (gauge invariant) curvature of the connection. This means that the action is a sum over plaquettes of a function of the excess return on the plaquette arbitrage operation as it was shown above.

Postulate 3: Free field theory – correspondence principle

We postulate that *the action is linear in the plaquette curvatures on the base graph*, since this is the simplest choice. It will be shown later that this postulate is equivalent to the free field theory description in the absence of matter fields and produces quasi-Brownian walks for exchange/interest rates in the continuous time limit. This means that this approach generalizes the standard constructions of mathematical finance. This fact serves as a correspondence principle.

Postulate 4: Extremal action principle

In a fully rational and certain economic environment it should not be possible to have "something from nothing", i.e. to have higher returns than the riskless rate of return. In a more general form, *the excess rate of return (rate of return above the riskless rate) on any kind of operation takes on the smallest possible value* which is allowed by the external economic environment. Together with the locality of the action, this give the extremality principle for the action.

Postulate 5: Limited rationality and uncertainty

The real environment *is not certain and not fully rational* and there exist nonzero probabilities to get different excess rates of return (exchange rates, prices and interest rates fluctuate and bear local arbitrage opportunities). We assume that the possibilities to have the excess return $R(x,T)$ at point of "space" x and time moment $T=0$ are statistically independent for different x, T, and are distributed with an exponential probability weight $e^{-\beta R(x,T)}$ with some effective measure of rationality β. If $\beta \to \infty$, we return to a fully rational and certain economic environment.

Formally, we state that the probability $P(\{U_{i,k}\})$ to find a set of exchange rates/ interest rates $\{U_{i,k}\}$ is given by the expression:

$$P(\{U_{i,k}\}) \sim e^{-\beta \sum_{(x,T)} R(x,T)} \sim e^{-\beta s_{gauge}} \; .$$

Now we are ready to write down the general action for the exchange/interest rate factors. However, before doing so, we would like to consider in more details a very simple example which gives some insight into the general framework. Let us, once again, consider two-asset systems (cash-shares).

The first thing to mention is gauge fixing. Since the action is gauge invariant, it is possible to perform a gauge transformation, which will not change the dynamics, but will simplify further calculations. In lattice gauge theory [9], there are several standard choices for fixing the gauge, and the axial gauge is one of them. In the axial gauge an element of the structural group is taken to be constant on links in the time direction (we keep them $e^{r_{1,2}\Delta}$), and one of the exchange rates (an element along the "space" direction at some particular chosen time) is also fixed. Below, we fix the price of shares at moment $T=0$ taking $S_0 = S(0)$. This means that in the situation of the ladder base the only dynamical variable is the exchange rate (price) as a function of time and the corresponding measure of integration is the invariant measure $\frac{\mathrm{d}S_i}{S_i}$.

From the above derivation, the definition of the distribution function for the exchange rate (price) $S = S(T)$ at the moment $T = N\Delta$ subject to the condition that at moment $T = 0$ the exchange rate was $S_0 = S(0)$ is given by:

$$P(0, S_0; T, S) =$$

$$\int_0^\infty \dots \int_0^\infty \prod_{i=1}^{N-1} \frac{\mathrm{d}S_i}{S_i} \exp\left[-\frac{\beta}{2\Delta} \sum_{i=0}^{N-1} \left(S_i^{-1} e^{r_2\Delta} S_{i+1} e^{-r_1\Delta} + S_i e^{r_1\Delta} S_{i+1}^{-1} e^{-r_2\Delta} - 2\right)\right] . \tag{5}$$

It is not difficult to see that in the limit $\Delta \to 0$ the expression in brackets converges to the integral

$$-\frac{\beta}{2} \int_0^T d\tau \left(\frac{\partial S(\tau)}{\partial \tau} / S(\tau) + (r_2 - r_1)\right)^2 ,$$

which corresponds to a geometrical random walk. Evaluating the integral and taking into account the normalization condition we come to the following expression for the distribution function of the price $S(T)$:

$$P(S(T)) = \frac{1}{\sigma S \sqrt{2\pi T}} e^{-(ln(S(T)/S(0)) - (\mu - \frac{1}{2}\sigma^2)T)^2/(2\sigma^2 T)} \tag{6}$$

Here we introduced the so-called volatility σ as $\beta = \sigma^{-2}$ and the average rate of share return μ as $\mu = r_1 - r_2$.

It is easy to give an interpretation to this last relation. The system as a whole is not conservative, and both the interest rate r_1 and the rate r_2 come from outside of the system (from banks and the performance of the corresponding company). Let us imagine that the world is certain, and that due to production performance, the value of the firm has increased. For this amount new shares with the same price S_1 have been issued (no dividends have been paid). The number of new shares for each old share is equal to $e^{r_2\Delta}$. This means that the cumulative (old) share will have a price $S_1 e^{r_2\Delta}$, while the original price (at zero moment) was S_0. Taking into account discounting and certainty, we end up with the following expression:

$$S_1 = e^{(r_1 - r_2)\Delta} S_0 ,$$

which tells us that the rate of return on the share is equal to $r_1 - r_2$. After introducing an uncertainty, this last expression turns into an average rate of return on the share.

Eq. (6) returns us to a justification of Postulate 3 which, together with other Postulates, is equivalent to a log-normal model for price walks in the absence of matter fields, which we consider in details in the next section.

Now we give the general expression for the probability distribution of a given exchange rate and internal rate of return profile:

$$P(\{S_{i,k}\},\{r_{i,k}\}) \sim exp\left(-\frac{\beta}{2\Delta}\sum_{plaquettes}\left(\prod_m U_m + \prod_m U_m^{-1} - 2\right)\right) \quad , \qquad (7)$$

where the sum is calculated over all plaquettes in the base graph with all links with nonzero connectivity Γ, the index m runs over all links of a plaquette, $\{U_m\}$ are corresponding elements of the structural group, which perform the parallel transport along the links.

To complete the gauge field consideration we want to return once again to the gauge invariance principle. The next section is devoted to the consideration of the "matter" field which interacts through the connection. It is clear now that the field represents a number of assets (as a phase in electrodynamics) and has to be gauged by the connection. Dilatations of money units (which do not change rules of investors behaviour) play the role of gauge transformation, which eliminates the effect of the dilatation by an appropriate tuning of the connection (interest rate, exchange rates, prices and so on), exactly as it is in the Fisher formula for the real interest rate in the case of inflation [2]. The symmetry with respect to a local dilatation of money units (security splits and the like) is the gauge symmetry of the theory.

2.4. EFFECTIVE THEORY OF CASH-DEBT FLOWS: MATTER FIELDS

Now let us turn our attention to "matter" fields. These fields represent cash-debt flows on the market. The importance of the cash-debt flows for our considerations is due to their role in the stabilization of market prices. Indeed, if, say, some bond prices eventually go down and create a possibility to get bigger returns than from other assets, then an effective cash flow appears, directed to these more valuable bonds. This causes an upward shift of the prices, due to the supply and demand mechanism. Altogether these effects smooth the price movements. The same picture is valid for debt flows, if there is a possibility for debt restructuring. As we will see all these features will find their place in our framework.

To formulate an effective theory for the flows we will assume that:

1. Any particular trader tries to maximize his return on cash and minimize his debts.
2. Traders' behaviour is *limitedly* rational i.e. there are deviations from pure rational strategy because of, for example, lack of complete information, specific financial manager's objectives [2] and so on.

We start with a construction of an effective theory for the cash flows and then generalize it to allow for the presence of debts. The first assumption tells us that an investor tries to maximize the following expression for the multiplier of the value of his investment (in the case of cash, shares and securities):

$$s(C) = ln(U_1 U_2 .... U_N)/\Delta$$

by a certain choice of the strategy which results in the corresponding trajectory in "space"-time for assets. Here $\{U_i\}_{i=i}^N$ are exchange (price) or interest factors which came from a choice of traders' behaviour at the i-th step on the trajectory C and boundary points (at times $T = 0$ and $T = N$) are fixed. We assume that there is a transaction time, which is the smallest time in the systems and is equal to Δ. In other words, a rational trader will choose the trajectory C_0 such that

$$s(C_0) = max_{\{C\}} s(C) \ .$$

Choosing the best strategy, a fully rational investor maximizes his return s. However, as we have assumed limited rationality, in analogy with the corresponding consideration of the connection field probability weights, we define the following probability weight for a certain trajectory C with N steps:

$$P(C) \sim e^{\beta' s(C)/\Delta} \equiv \ e^{\beta s(C)}, \tag{8}$$

with some "effective temperature" $1/\beta$ which represents a measure of the average irrationality of the traders per unit time.

It is possible to generalize the approach to a case of many investors operating with cash and debts. The corresponding functional integral representation for the transition probability (up to a normalization constant) has the form [1]:

$$\int D\psi_1^+ D\psi_1 D\psi_2^+ D\psi_2 D\chi_1^+ D\chi_1 D\chi_2^+ D\chi_2 e^{\beta(A+A')} \ ,$$

with the actions for cash and debt flows:

$$A = \frac{1}{\beta} \sum_i (\psi_{1,i+1}^+ e^{\beta r_1 \Delta} \psi_{1,i} - \psi_{1,i}^+ \psi_{1,i} + \psi_{2,i+1}^+ e^{\beta r_2 \Delta} \psi_{2,i} - \psi_{2,i}^+ \psi_{2,i}$$

$$+ (1-t)^\beta S_i^\beta \psi_{1,i+1}^+ \psi_{2,i} + (1-t)^\beta S_i^{-\beta} \psi_{2,i+i}^+ \psi_{1,i}) \ , \tag{9}$$

$$A' = \frac{1}{\beta} \sum_i (\chi_{1,i+1}^+ e^{-\beta r_1 \Delta} \chi_{1,i} - \chi_{1,i}^+ \chi_{1,i} + \chi_{2,i+1}^+ e^{-\beta r_2 \Delta} \chi_{2,i}$$

$$- \chi_{2,i}^+ \chi_{2,i} + (1+t)^{-\beta} S_i^{-\beta} \chi_{1,i+1}^+ \chi_{2,i} + (1+t)^{-\beta} S_i^\beta \chi_{2,i+1}^+ \chi_{1,i}) \ . \tag{10}$$

We do not stop here to describe the boundary conditions, which are discussed in details in [1, 10].

It is interesting to note that Eq. (10) may be transformed to Eq. (9) by inverting the signs of r_i and inverting exchange rates S in the absence of transaction costs. This corresponds to the transformation from negative to positive charges. The transaction costs make this symmetry only approximate.

In the absence of restructuring of debts (i.e. one kind of debts cannot be transformed into another kind of debt) the last terms containing S have to be cancelled. Then the action takes the particularly simple form:

$$A'_0 = \frac{1}{\beta}\sum_i(\chi^+_{1,i+1}e^{-\beta r_1\Delta}\chi_{1,i} - \chi^+_{1,i}\chi_{1,i} + \chi^+_{2,i+1}e^{-\beta r_2\Delta}\chi_{2,i} - \chi^+_{2,i}\chi_{2,i}) \ . \quad (11)$$

Everything that we have said above was related to the particular case of the two assets problem. Now we give a form of the action for the most general situation. As it was shown before, a general system of assets is described by a form of the "space"-time base graph L of the fibre bundle. Elements of the graph are labeled by (i,k), where the index i, $i = 0,..,N$, represents a particular asset (a point in "space") and the index k labels time intervals. By the definition of the base L any two points of the base can be linked by a curve γ, each formed by elementary segments with a nonzero connectivity Γ. On its own, each of the segments is provided with an element of the structural group U, which performs a parallel transport along this (directed) link. These allow us to give the most general form of the action A $((i_{12},k_{12}) \equiv (i_1,k_1),(i_2,k_2) \in L : \Gamma((i_1,k_1),(i_2,k_2)) = 1)$:

$$\begin{aligned} A &= \frac{1}{\beta}\sum_{(i_{12},k_{12})}(\psi^+_{(i_1,k_1)}U^{\beta}_{(i_{12},k_{12})}[1 - t(1-\delta_{i_1,i_2})]^{\beta}\psi_{(i_2,k_2)} \\ &- \delta_{(i_1,k_1),(i_2,k_2)}\psi^+_{(i_1,k_1)}\psi_{(i_1,k_1)}) \ . \end{aligned} \quad (12)$$

In the same way, the action for the debt flows can be written:

$$\begin{aligned} A' &= \frac{1}{\beta}\sum_{(i_{12},k_{12})}(\chi^+_{(i_1,k_1)}U^{-\beta}_{(i_{12},k_{12})}[1 + t(1-\delta_{i_1,i_2})]^{-\beta}\chi_{(i_2,k_2)} \\ &- \delta_{(i_1,k_1),(i_2,k_2)}\chi^+_{(i_1,k_1)}\chi_{(i_1,k_1)}) \ . \end{aligned} \quad (13)$$

3. Conclusion

We have proposed a mapping of Capital Market Theory onto Lattice Quantum Gauge Theory, where the gauge field represents the interest rate and prices and "matter" fields are cash-debts flows. Based on the mapping, we have derived action functionals for both the gauge field and "matter" fields assuming several postulates. The main assumption is the gauge invariance of the dynamics which means that the dynamics does not depend on particular values of units of assets, and a change of the values of the units may be compensated for by a proper change of the gauge field. The developed formalism has been applied to some issues of Capital Market Theory. Thus, it was shown in [10] that a deviation of the distribution function from the log-normal distribution may be explained by an active trading behavior of arbitragers. In this framework, such effects as changes in shape of the distribution function, "screening" of its wings for large values of the price and the non-Markovian character (memory) of price random walks turned out to be consequences of the damping of the arbitrage and directed price movements caused by speculators. In [11] consequences of the bid-ask spread and the corresponding gauge invariance breaking have been examined. In particular, it turned out that the distribution function is also influenced by the bid-ask spread and the change of its form may be explained by this factor as well. So, the complete analysis of the statistical characteristics of prices have to account for both these factors. Moreover, it is possible to show [12] that the Black-Scholes equation for financial derivatives can be obtained in the present formalism in the absence of speculators (i.e. absence of the arbitrage game).

Let us now make two final remarks.

1. In all the above cited references a very simple model of the stock exchange was considered, where only one kind of security is traded. This can be generalized to a more realistic situation with a set of traded securities. Following this line, dynamical portfolio theory can be constructed and, in the static (equilibrium) limit, it will coincide with standard portfolio theory [13]. In the dynamical theory time-dependent correlation functions will play the role of response functions of the market to an external perturbation, such as a new information or a change in the macroeconomic environment. Taking into account virtual arbitrage fluctuations will lead to a time-dependent modification of CAPM or ARBM [3].
2. Since the influence of the speculators leads to the non-Markovian character of price walks, there is no possibility to eliminate risk using arbitrage arguments to derive an equation for the price of a derivative. Then, virtual arbitrage and corresponding asset flows have to be considered, leading to a correction of the Black-Scholes equation.

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