HIERARCHICAL STRUCTURES IN FINANCIAL MARKETS: A "TURBULENT APPROACH" TO STOCHASTIC VOLATILITY

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1. Introduction

Price fluctuations in financial markets are random, but not Gaussian. Closer analysis reveals more intricate structures. Understanding them is important for both forecasting and risk analysis. However, up to now it is not clear how to characterize these structures.

There is a huge econometric literature on how to model financial time series. The main approach consists in modelling the stochastic properties but there are also models that grow from the market structure.

Another approach consists in using analogies with phenomena in other fields, physics in particular. Frequently, physics has served as a reference for explaining phenomena in other fields. An analogy with a phenomenon in physics may turn out to be very fruitful, but one should bear in mind the limits of any analogy, considering the pecularities of economic phenomena.

In modern physics, the study of collective phenomena has become increasingly important. Keywords are phase transitions, pattern formation, self-organized criticality, and hydrodynamic turbulence. The corresponding theories are quite different from those belonging to the 19th century mechanistic framework. From a fundamental point of view the main characteristic of collective phenomena is the emergence of new behaviour which is not displayed by the constituent parts of the system and cannot be deduced by simple aggregation; it is essentially due to the interaction of the parts. Often the new features are largely independent of the nature of the parts of the system, and nonlinearities play an essential role. Theories describing this kind of behaviour are likely to be also useful in the social sciences.

Frequent findings in complex systems displaying collective behaviour are scaling laws, which show up for quite different phenomena. They essentially state the absence of a natural scale. In economics, they have first been observed for fluctuations of cotton prices [1]. Later they were also found to hold for other financial instruments such as foreign exchange (FX) rates [2, 3] and stock exchange indices [4]. Scaling laws may have different origins but often hierarchical (or self-similar) structures are at their root. In any case, one should try to find the underlying structural (or dynamic) factors and work out more specific consequences of such factors.

The assumption underlying the hypothesis presented in this paper is that the pricing process in financial markets is a hierarchically structured collective phenomenon. It is argued that this process shares important features with fully-developed hydrodynamic turbulence. In hydrodynamic turbulence there is a (nearly) self-similar hierarchy of vortices caused by a flow of energy from large to small spatial scales. It is this cascade that gives rise to the scaling behaviour of the moments of the distribution of velocity fluctuations in a turbulent flow. Another consequence of the energy cascade is that the distribution of differences of the velocities at two points in the flow is not Gaussian and its shape changes when the distance between the points is varied. This distribution can be represented as a mixture of Gaussians, and the Gaussian components can be recovered from the experimental data.

The essence of the analogy with turbulence is that in some financial markets such as the FX market, there is a cascade from long to short temporal scales similar to the energy cascade in a turbulent flow [5]. As in turbulence, this cascade does not only account for the scaling behaviour of the moments of the return distribution but also its change in shape as a function of the time delay. This distribution has been represented as a mixture of Gaussians, and the Gaussian components have been (approximately) recovered from the series of quotes [6]. Following the hypothesis proposed by Ghashghaie et al. [5] of a cascade across the temporal scales in the FX market, Arneodo et al. [7] visualized a cascade for stock market data and showed that the multiplicative cascade-model implied by the analogy with turbulence reproduces the slowly decaying volatility autocorrelation function well.

The outline of this paper is as follows. We start Sec. 2 with an account of the most important facts about the FX market. A description of the main characteristics of FX data follows. Sec. 3 deals with basic facts about hydrodynamic turbulence, in particular those concerning the statistical behaviour of velocity differences and their relation to the underlying energy cascade. In the second part of this section, some similarities between turbulence and price dynamics in the FX market are reviewed. In Sec. 4, two implications of the hierarchical structures in the FX market are presented, and in the final section we end with a discussion.

2. The structure of the foreign exchange market

2.1. ORGANISATION OF THE MARKET

Since the breakdown of the Bretton Woods agreement in the early seventies, the major exchange rates have been determined by the market and, consequently, they are constantly fluctuating. Since that time, different regional markets have been growing together and nowadays they form a single global market the participants of which are connected via computer networks. The market is active practically round-the-clock with a short gap between the close down of the markets on the west coast in the US and the opening of those in the Far East. Trading is interrupted only during the weekends and world wide holidays. Traditionally, deals were executed via telephone. In the recent past, computer trading has been gaining ground at the expense of the telephone.

In the FX market there is practically no private information. The liquidity is very large: the volume traded per day amounts to more than 10^{12} US-dollars (approximately the GNP of Italy). Typical sizes of individual deals are of the order of 10^6 to 10^7 US dollars.

Unfortunately, volume data is scarce. (Some of the rare studies with FX volume data are found in [8, 9, 10].) The quotes, which are distributed via computer networks, can be easily collected, but they are only indicative prices, which may differ from prices at which the deals are actually executed. Comfortingly, Evans [10] found that price quotes and trading activitiy are closely related, even though quote intensity will typically understate trade intensity in busy market conditions.

Our analysis has been performed with a data set collected by Olsen & Associates[1] containing the FX quotes of the spot rates for the three major currencies in the period from October 1, 1992 till September 30, 1993.

2.2. STATISTICAL PROPERTIES OF PRICE SERIES

2.2.1. *Stochastic volatility models*

The statistical properties of market prices began to interest scientists around the turn of the century. A pioneering work is the thesis of Louis Bachelier [11] who developed the theory of Brownian motion while investigating the price fluctuations on the Paris stock exchange.

[1]Olsen & Associates, Seefeldstraße 233, Zürich, Switzerland

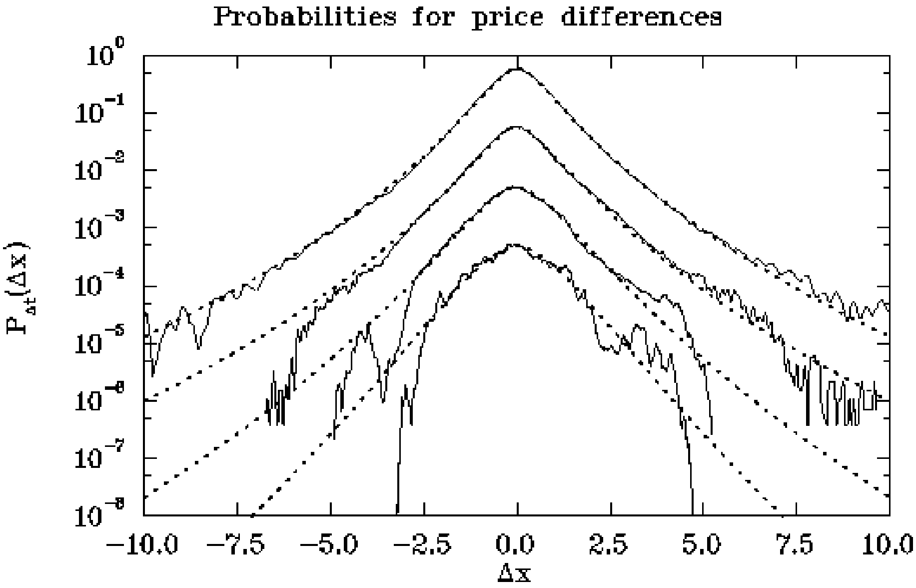

Figure 1. *Pdf*s for USD–DEM middle prices with $\Delta t \approx$ 10min, 1.5h, 11.5h, 46h (from top to bottom). Full lines: observed data. Dashed lines: results of (least squares) fits carried out with the mixture of distributions (1). For better visibility the curves have been shifted vertically with respect to each other.

The basic assumption of the early works was that the prices perform a Gaussian random walk. Transferred to the logarithmic prices, this has remained the main paradigm till recently (work of Black and Scholes [12]). However, it is not supported by empirical observations. Observed distributions of FX and asset returns[2] Δx over a time interval Δt typically exhibit heavy tails, see Fig. 1. This effect is more pronounced for large Δt than for small Δt.

Another important aspect is volatility clustering, i.e., the occurrence of agitated periods with large price fluctuations and of quiet periods with low price fluctuations. This implies correlations between the volatility at different times. The corresponding correlation function decays very slowly [13].

There is a huge econometric literature dealing with the statistical behaviour of price series in financial markets. Among the major approaches to time series modelling are the stochastic volatility models [14], which assume the variance of returns to be a time dependent stochastic process and the conditional distribution of returns (given the instantaneous volatility) to be Gaussian. These include in particular subordinated models (e.g. [15])

[2]For a series of (middle) prices $p(t)$, the returns Δx are defined as logarithmic price changes over a time interval Δt, $\Delta x = \log p(t + \Delta t) - \log p(t)$

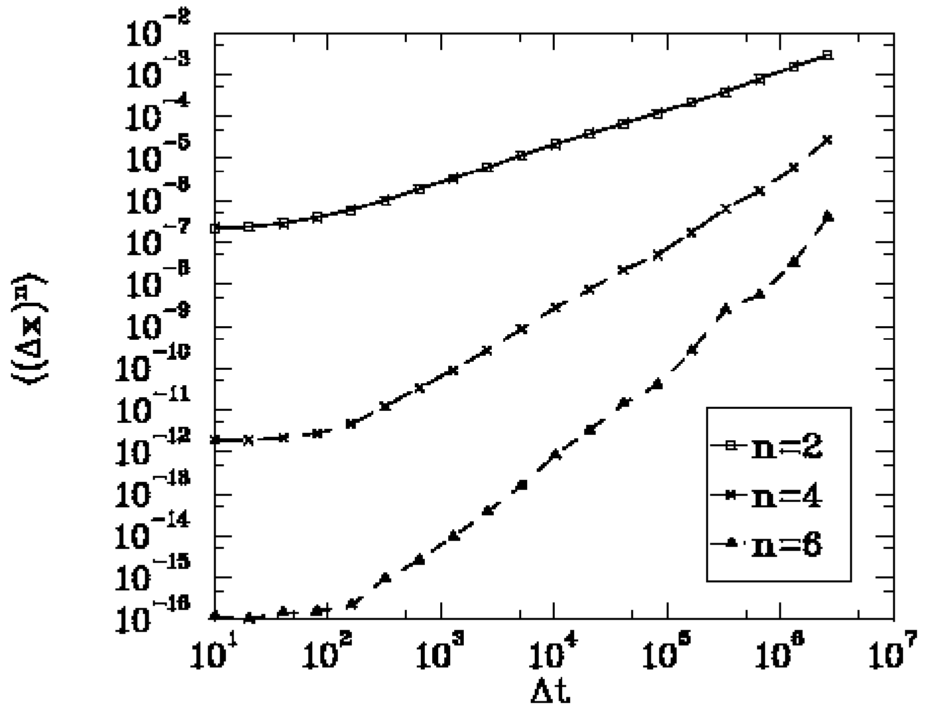

Figure 2. Log-log plot of the moments $\langle(\Delta x)^n\rangle$ of order n as function of the time delay Δt.

and information-counting models (e.g. [16]). ARCH models [17] and generalizations [18, 19] consider the volatility as an autoregressive process.

In the framework of stochastic volatility models, the *pdf* $p_{\Delta t}(\Delta x)$ of returns Δx over a time interval Δt is often described by a mixture of Gaussians with lognormally distributed variances (cf., e.g., [15]):

$$p_{\Delta t}\left(\Delta x\right) = 3D \int_{-\infty}^{\infty} h_{\Delta t}(\log\sigma;\lambda^2) p_0\left(\Delta x;\sigma^2\right) d(\log\sigma), \qquad (1)$$

where $h_{\Delta t}(\log\sigma;\lambda^2)$ is the *pdf* of $\log\sigma$ with variance λ^2. An extensive generalization of mixture representations has been proposed by Barndorff-Nielsen [20, 21].

Currently, the time variation of the volatility is a generally accepted feature. In order to account for the slow decay of the volatility autocorrelation function, however, one has to take into account different time scales, as has been done by Müller et al. [13] in the framework of a heterogenous ARCH approach (HARCH).

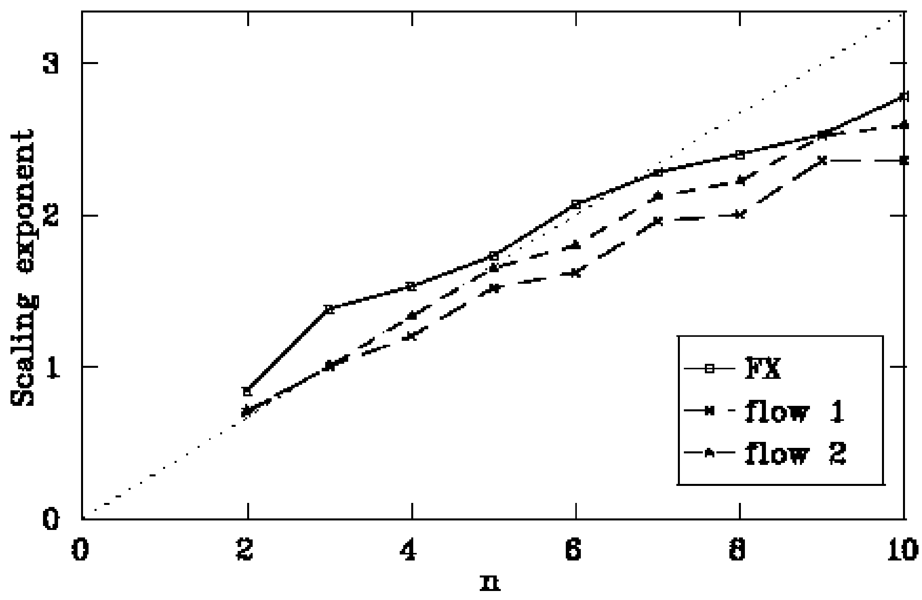

Figure 3. The n dependence of the scaling exponents ξ_n and ζ_n for the nth moments of the distribution of FX price changes (squares), and of the distribution of velocity differences in turbulent flows, taken from [22] (crosses) and [23] (triangles). Note the same qualitative deviation of all curves from a straight line.

2.2.2. *Scaling properties*

As already pointed out in the introduction, scaling properties are quite common in economic data sets. In financial price series, scaling behaviour has been found for the nth moments $\langle(\Delta x)^n\rangle(\Delta t)$ of the returns Δx as function of the time delay Δt (see Fig. 2):

$$\langle(\Delta x)^n\rangle(\Delta t) = (\Delta t)^{\xi_n} \tag{2}$$

The value of the exponent ξ_1 reported in [5] for US dollar–Deutsch mark FX rates is about 0.45. Other authors obtained other values, e.g., $\xi_1 = 0.59$ [2]. A detailed study revealed [24] that the values of the exponents ξ_1 and ξ_2 vary by about 30%, depending on the method used for their evaluation. For the exponents of higher-order moments, the uncertainties become even larger. Thus few conclusions can be drawn from the value of a single exponent alone. In all cases, however, ξ_n as a function of n is a concave function (cf. Fig. 3) [5], in contrast to a random walk for which $\xi_n = n/2$. The analysis can be extended to non-integer exponents and the tools of multifractal analysis can be used. A corresponding study [25] confirmed the concave shape of the scaling exponents (vs. n) for FX data.

2.2.3. *Change in shape of the distribution of returns*
If the scaling exponent of the nth moment depended linearly on n, the shape of the distribution of returns would not depend on the time delay. The concave form of the ξ_n vs. n graph implies that the distribution changes its shape as a function of Δt. Indeed, the distribution is leptocurtic for short time delays and becomes more and more Gaussian-shaped for larger and larger time delays (see Fig. 1).

This change in shape is not explicitly addressed in the traditional econometric approaches, which mainly deal with fixed time delays Δt. A GARCH(1,1) model, e.g., only partly accounts for this behaviour, as follows from the results of Mantegna and Stanley [4]. They propose an alternative approach, which consists in assuming a random walk with price changes following a truncated Lévy distribution [4, 26]. Owing to the truncation, the variance is finite, which ensures asymptotic normality of the price changes for large time delays. This approach is appealing because of its simplicity, and it models quite nicely the change in shape of the return distribution as a function of the time delay. Its main shortcoming, however, is that it cannot account for the empirically observed clustering of volatility. This is a step back with respect to the stochastic volatility models in econometrics.

3. Hydrodynamic turbulence: an analogy from physics

3.1. STATISTICAL PROPERTIES OF TURBULENT FLOW

Hydrodynamic turbulence [27, 28, 29, 30, 31, 32] is a widespread phenomenon. Examples are the weather, which, in the form of cyclones, produces vortices of the order of hundreds of kilometers, the wake behind all kind of objects moving in a gas or a liquid, or the flow behind an obstacle. The main characteristic of these phenomena is a hierarchy of vortices [33], which provides a mechanism for dissipating large amounts of energy in a viscous fluid. The qualitative aspect of nested vortices of different sizes was already represented in drawings by Leonardo da Vinci. The idea that energy is pumped into the system at a large scale, transferred by the vortices to smaller and smaller scales and dissipated at the smallest scale, is due to Richardson [34]. This qualitative picture of the flow of energy across scales has been cast into a quantitative model by Kolmogorov [29, 30] and Obukhov [31]. Their model is based on reasonable ad hoc assumptions; a rigorous derivation from the Navier-Stokes equations has not yet been possible. In the spirit of the 1962 version of Kolmogorov's model, one can assume that, at level[3] i, the energy dissipation rate ϵ_i of a vortex is obtained from the energy dissipation rate ϵ_{i-1} of the vortex at level $i-1$ by

[3] The number counting the levels can be interpreted as wave number.

multiplication with a random factor a_i with a mean value less than 1:

$$\begin{aligned}\epsilon_i &= a_i\epsilon_{i-1} \\ &= a_i a_{i-1}\epsilon_{i-2} \\ &= a_i a_{i-1}\cdots a_1\epsilon_0.\end{aligned} \quad (3)$$

This means that the logarithm of ϵ_i is related to the logarithm of ϵ_0 by a sum of random variables $\sum_k \log a_k$ which are assumed to be independent and of finite variance. Thus, for a sufficiently large i, the energy dissipation rate ϵ_i is lognormally distributed.

The flow of energy across the vortices at different scales induces a scaling behaviour of the nth moments $\langle(\Delta v)^n\rangle(\Delta r)$ of the velocity difference Δv as a function of the distance Δr between two points at which the velocities are measured: $\langle(\Delta v)^n\rangle(\Delta r) \propto (\Delta r)^{\zeta_n}$. In Kolmogorov's 1962 model, the scaling exponents are given by $\zeta_n = n/3 - \mu(n-3)n/18$, which are concave functions with $\zeta_3 = 1$. For experimental results, see Fig. 3.

In practice, the evaluation of higher order moments is affected by large errors. Therefore, it is advantageous to study directly the distribution of velocity differences. Barndorff-Nielsen [20] has shown that this distribution can be well represented as mixtures of distributions of the same type. Chabaud et al. [35] obtained similar results by a mixture of Gaussians (1) with lognormally distributed variances. The variance $\lambda^2 = \lambda^2(\Delta r)$ of the lognormal distribution is a shape parameter which vanishes for large Δr. A hierarchical coupled maps model proposed by Beck [36] also reproduces well the shape of the distributions of velocity differences.

3.2. SIMILARITIES BETWEEN VELOCITY FLUCTUATIONS AND RETURNS

Returns in financial markets such as the FX market exhibit similar statistical properties as velocity fluctuations in a turbulent flow. In particular, the distributions of both velocity fluctuations and returns deviate from a Gaussian distribution in a similar way (both display fat tails and a pronounced peak) and change their shape as a function of the scale variable (time delay for prices and spatial distance for velocities). The moments of both distributions display scaling behaviour. Even though the values of the scaling exponents ξ_n and ζ_n are different, their functional dependence on n is similar: both form a concave curve, see Fig. 3.

As pointed out above, mixtures of Gaussians with lognormally distributed variances are used both in physics and economics to represent distributions of velocity fluctuations in a turbulent flow and of price changes in financial markets, respectively. The difference, however, is that (as men-

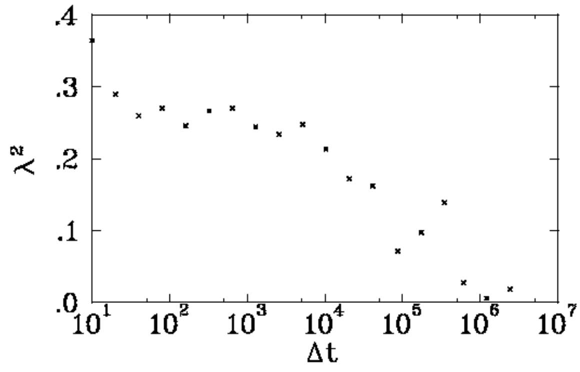

Figure 4. Variation of the shape parameter λ^2 as function of Δt.

tioned in Sec. 2) traditionally in econometrics only fixed time delays were considered.

Adjusting the shape parameter $\lambda^2 = \lambda^2(\Delta t)$, Ghashghaie et al. [5] have shown that for US dollar–Deutsch mark spot rates, the change in shape of the *pdf* of returns can be described by the mixture (1). For time delays ranging from 10 minutes up to about 2 days, λ^2 depends approximately linearly on $\log \Delta t$ (Fig. 4). This is consistent with Kolmogorov's 1962 model of turbulence. The shape of the distributions of returns has also been reproduced by Hilgers and Beck [37] using the same type of coupled maps model as in the case of turbulence.

4. Hierarchical features in FX markets

The similarities between the statistical properties of velocity fluctuations in turbulent flows and the FX returns mentioned in the previous section has prompted us to postulate the existence of a cascade in the price dynamics of the FX market similar to the energy cascade in turbulence. The large volume of inter-dealer transactions (about 75% . . . 90% of the whole trading volume) is a hint pointing in this direction. One may envisage that the internal dynamics of the cascade is due to inventory effects together with information arrival processes. Dealers try to avoid large open positions, especially overnight. Thus, a dealer having received an order of large size will try to make a compensatory deal. Since large buying or selling orders contain information for insiders, he may break large trades into smaller ones, thus conveying information to the market only gradually. Subsequent dealers may do the same, dividing an initially large order into still smaller ones and creating a kind of risk or information cascade.

The other ingredient is provided by external information reaching the market. Andersen and Bollerslev [38] have shown that heterogenous information may generate persistence in volatility and that the information dissemination processes may exhibit certain hierarchical features. They have also demonstrated how heterogenous information arrival processes with different decay times can explain scaling properties.

4.1. CORRELATION OF VOLATILITIES

In hydrodynamics, the picture of an energy cascade across spatial scales is widely accepted [33]. In financial markets, it is not a priori clear what the most adequate analogue of the energy flow could be. Certainly risk and information play an important role. Both are related to volatility. Therefore the volatility is a good condidate to start with in a quantitative cascade model. The effect of the asymmetry (from long to short Δt) of the information flow between market components with different time horizons was first shown by Müller et al. [13] in the intra day dynamics of the FX market. Later, Arneodo et al. [7] visualized the information flux across scales for stock market data.

An important feature, which is directly related to volatility clustering, is the slowly decreasing autocorrelation of volatilities. This feature can be naturally accounted for by the basic feature of the Kolmogorov model, namely the multiplicative cascade (3), where the ϵ_i's have to be interpreted as volatilities on the respective time scales. Arneodo et al. have shown [7] that the assumption of a multiplicative volatility cascade yields a logarithmic decay for the volatility autocorrelation function, which fits the observed decay well (even though an algebraic decay would do equally well). They also show that the correlation function predicted by the cascade model does not depend on the scale on which this function is calculated. This independence of scale is confirmed by the observed data.

4.2. FLUCTUATION INTENSITY AND CONDITIONAL RETURNS

If the multiplicative cascade (3) captures the essential structural and/or dynamic features, it should be possible to recover the Gaussian components $p_0(\Delta x; \sigma^2)$ of the mixture (1) by conditioning the returns on an adequate variable. In turbulence, such a decomposition has been successfully carried out by conditioning the velocity differences on the energy transfer rate [39]. The same kind of analysis has been performed for FX data [6] where a local measure of the intensity $I_{\Delta t}(t_i)$ of price fluctuations at time t_i (on the scale

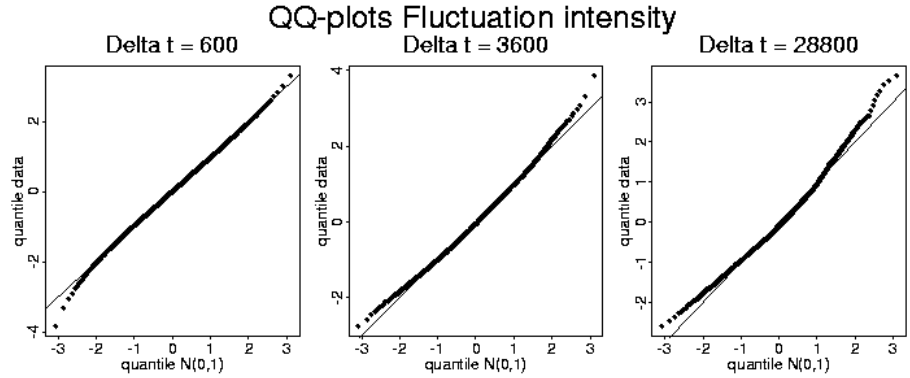

Figure 5. QQ-plots (versus a standard Gaussian) of the distribution of the logarithmic fluctuation intensity $\log I_{\Delta t}$ for $\Delta t =$ (bottom) 10 min., (middle) 1 hour, (top) 8 hours.

of the time delay Δt) has been defined as the conditioning variable [6]:

$$I_{\Delta t}\left(t_i\right) = \frac{1}{N(t_i,\Delta t)} \sum_{t_i<\tau_j\leq t_i+\Delta t} \left(x(\tau_j) - x(\tau_{j-1})\right)^2 - \left(\frac{x(t_i) - x(t_i+\Delta t)}{N(t_i,\Delta t)}\right)^2 \tag{4}$$

with the normalisation constant

$$N(t_i,\Delta t) = \sum_{t_i<\tau_j\leq t_i+\Delta t} 1. \tag{5}$$

This (modified) volatility can be interpreted as an information measure. As expected from the analogy with turbulence, $I_{\Delta t}$ is approximately log-normally distributed (Fig. 5), and the conditional distribution of returns given $I_{\Delta t}$ is approximately Gaussian (see Fig. 6). This corroborates the assumption that the cascade in the FX market can be described, at least in a first approximation, by means of volatilities. The conditional returns still deviate slightly from a Gaussian distribution. Alternative definitions of $I_{\Delta t}$ or other (volume based) quantities may yield better results.

5. Discussion

The facts reviewed in this paper show that there is clear evidence of a cascade from long to short time scales in the price dynamics of financial markets. Currently the most suitable variable for describing the flow across the time scales seems to be the volatility. This view is supported by numerical results obtained for large data sets. It is also consistent with the

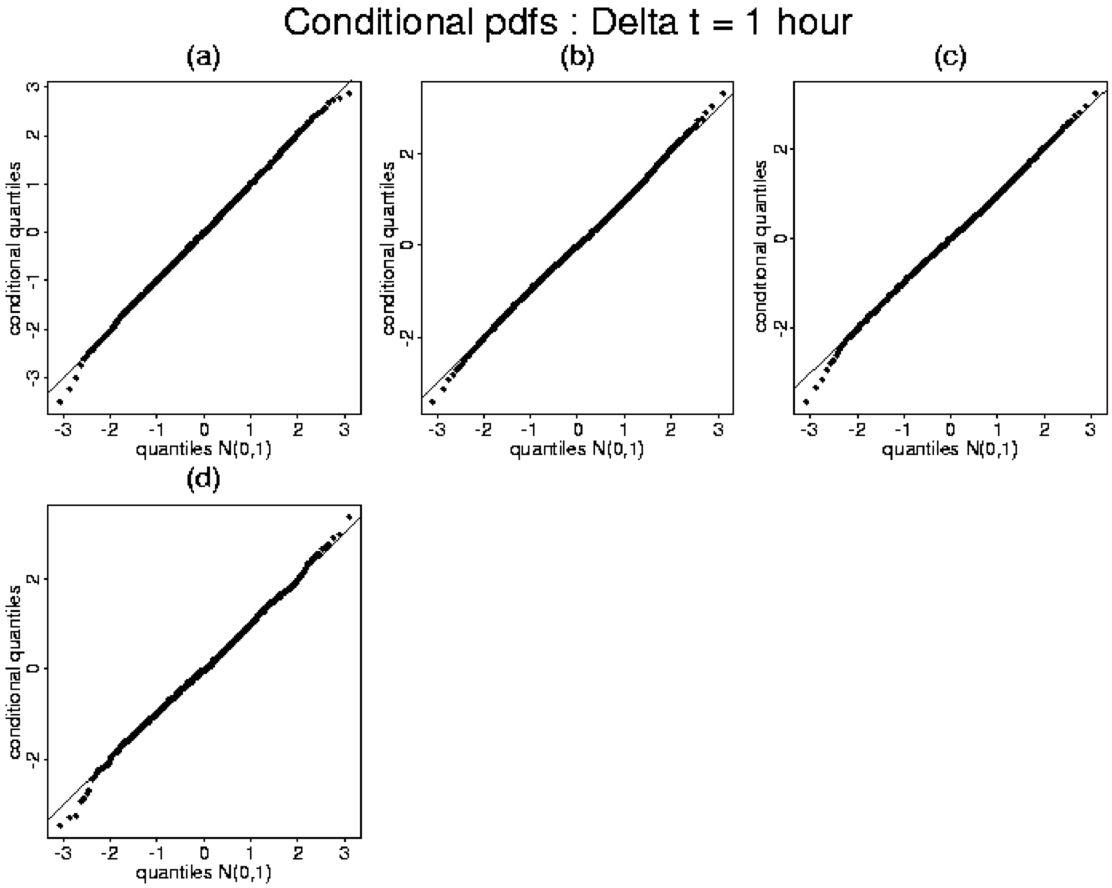

Figure 6. QQ-plots (versus a standard Gaussian) of the conditional distribution of returns for $\Delta t = 10$ minutes. (a) $\log I_{\Delta t} \in [-8.4, -8.1]$, (b) $\log I_{\Delta t} \in [-9.3, -9.0]$, (c) $\log I_{\Delta t} \in [-10.2, -9.9]$, (d) $\log I_{\Delta t} \in [-11.0, -10.8]$.

qualitative picture that the cascade is closely related to the information-processing activity of the market, and that inventory effects accounted for by the risk aversion of the dealers play an important role.

The information contained in the structure of the cascade can be used to improve risk management. To this end it is necessary to integrate the concept of a cascade into time series models for the return process. Possible approaches are those pursued in economics to account for information and inventory effects on the prices. Hierarchical models inspired by the Kolmogorov cascade, such as coupled maps models [37], may serve as alternative starting points.

We would like to close with a general remark. The link between models growing from market microstructure and time series models reproducing the stochastic properties of the return process is of the same qualitative nature as the link between the Navier-Stokes equations, which describe the motion of a viscous fluid, and stochastic turbulence models describing the statistics of velocity fluctuations in a turbulent flow. The situation in hydrodynamics is still unsatisfactory in so far as it has not been possible to derive a turbulence model from the Navier-Stokes equations. Additional assumptions are necessary. In economics the situation is far less satisfactory because, in contrast to the Navier-Stokes equations, no precise "microscopic laws" governing market dynamics are known. However, even if there were such laws this would perhaps be of little help. The analogy with the physical phenomenon suggests that it is questionable whether the statistical properties of the price dynamics of financial markets could be derived from such "microscopic laws" without additional assumptions about the stochastic nature of the return process.

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