Class 9 - Multipole radiation, scattering

Class material

Exercise 9.1 - Rotating charge systems (Jackson 9.1)

A common textbook example of a radiating system is a configuration of charges fixed relative to each other but in rotation. the charge density is obviously a function of time, but it is not in the from of

$$\rho(\mathbf{x},t) = \rho(\mathbf{x})\,\mathrm{e}^{-\imath\omega t} \ . \tag{1}$$

(a) Show that for rotating charges one alternative is to calculate *real* time-dependent multipole moments using $\rho(\mathbf{x},t)$ directly and then compute the multipole moments for a given harmonic frequency with the convention of (1) by inspection or Fourier decomposition of the time-dependent moments. Note that care must be taken when calculating $q_{lm}(t)$ to form linear combinations that are real before making the connection.

(b) Consider a charge density $\rho(\mathbf{x},t)$ that is periodic in time with period $T = 2\pi/\omega_0$. By making a Fourier series expansion, show that it can be written as

$$\rho(\mathbf{x},t) = \rho_0(\mathbf{x}) + \sum_{n=1}^{\infty} \mathrm{Re}\left[2\rho_n(\mathbf{x})\,\mathrm{e}^{-\imath n\omega_0 t}\right]$$

where

$$\rho_n(\mathbf{x}) = \frac{1}{T}\int_0^T \rho(\mathbf{x},t)\,\mathrm{e}^{\imath n\omega_0 t}\,\mathrm{d}t$$

This shows explicitly how to establish connection with (1).

(c) For a single charge q rotating about the origin in the $x-y$ plane in a circle of radius R at constant angular speed ω_0, calculate the $l=0$ and $l=1$ multipole moments by the methods of part (a) and (b) and compare. In method (b) express the charge density $\rho_n(\mathbf{x})$ in cylindrical coordinates. Are there higher multipoles, for example, quadrupole? At what frequencies?

Exercise 9.2 - Radiation of two half-spheres with oscillating potential difference (Jackson 9.3)

Two halves of a spherical metallic shell of radius R and infinite conductivity are separated by a very small insulating gap. An alternating potential is applied between the two halves of the sphere so that the potentials are $\pm V\cos\omega t$. In the long-wavelength approximation limit, find the radiation fields, the angular distribution of radiated power, and the total radiated power from the sphere.

Exercise 9.3 - Scattering on a perfectly conducting sphere

Consider a small perfectly conducting sphere of radius a, and an unpolarized incident light wave of wave-length much larger than a.

(a) Using the perfect conductor limits of the dielectric sphere in external homogeneous electric field, compute the electric dipole polarisation of the sphere.

(b) Using the perfect conductor limits of the magnetic sphere in external homogeneous magnetic field, compute the magnetic dipole polarisation of the sphere.

(c) Determine the differential cross section amd the polarisation of the outgoing wave.

Exercise 9.4 - Radiation of a current loop (Jackson 9.14)

An antenna consists of a circular loop of wire of radius a located in the $x-y$ plane with its center at the origin. The current in the wire is

$$I = I_0 \cos \omega t = \mathrm{Re}\, I_0\, \mathrm{e}^{-\imath \omega t}$$

(a) Find the expression for $\mathbf{E}$ and $\mathbf{H}$ in the radiation zone without approximations as to the magnitude of ka. Determine the power radiated per unit solid angle.

(b) What is the lowest nonvanishing multipole moment (Q_{lm} or M_{lm})? Evaluate this moment in the limit $ka \ll 1$.

Homework

The following problems (marked with an asterisk) form the basis of the short test at the beginning of the next class.

Exercise 9.5 - Scattering of polarized light on a perfect conducting sphere (Jackson 10.1)*

(a) Show that for arbitrary initial polarization, the scattering cross section of a perfectly conducting sphere of radius a, summed over outgoing polarizations, is given in the long-wavelength limit by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\mathbf{e_0}, \mathbf{n_0}, \mathbf{n}) = k^4 a^6 \left[\frac{5}{4} - |\mathbf{e_0} \cdot \mathbf{n}|^2 - \frac{1}{4} |\mathbf{n} \cdot (\mathbf{n}_0 \times \mathbf{e}_0)|^2 - \mathbf{n}_0 \cdot \mathbf{n} \right]$$

where $\mathbf{n}_0$ and $\mathbf{n}$ are the directions of the incident and scattered radiations, respectively, while $\mathbf{e_0}$ is the (perhaps complex) unit polarization vector of the incident radiation ($\mathbf{e}_0^* \cdot \mathbf{e_0} = 1$; $\mathbf{n}_0 \cdot \mathbf{e}_0 = 0$).

(b) If the incident radiation is linearly polarized, show that the cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}(\mathbf{e_0}, \mathbf{n_0}, \mathbf{n}) = k^4 a^6 \left[\frac{5}{8}(1 + \cos^2\theta) - \cos\theta - \frac{3}{8} \sin^2\theta \cos 2\phi \right]$$

where $\mathbf{n} \cdot \mathbf{n}_0 = \cos\theta$ and the azimuthal angle ϕ is measured from the direction of the linear polarization.

(c) What is the ratio of scattered intensities at $\theta = \pi/2,\ \phi = 0$ and $\theta = \pi/2,\ \phi = \pi/2$? Explain physically in terms of the induced multipoles and their radiation patterns.

Exercise 9.6 - Radiation of a rotating quadrupole (Jackson 9.2)*

A radiating quadrupole consists of a square of side a with charges $\pm q$ at alternate corners. The square rotates with angular velocity ω about an axis normal to the plane of the square and through its center. Calculate the quadrupole moments, the radiation fields, the angular distribution of the radiation, and the total radiated power, all in the long-wavelenth approximation. What is the frequency of the radiation?

Exercise 9.7 - Radiation of an imperfect sphere (Jackson 9.12)*

An almost spherical surface defined by

$$R(\theta) = R_0[1 + \beta P_2(\cos\theta)]$$

has inside of it a uniform volume distribution of charge totaling Q (here P_2 denotes the Legendre polynomial of degree 2). The small parameter β varies harmonically in time at frequency ω. This correspondes to surface waves on a sphere. Keeping only lowest order terms in β and making the long-wavelength approximation, calculate the nonvanishing multipole moments, the angular distribution of radiation, and the total power radiated.

Hint: note that the leading multipole moment is the electric quadrupole, which is in fact the only nonvanishing multipole moment. Using spherical multipoles greatly simplifies the calculation of the quadrupole tensor.

These problems are for further practice and to have some fun!

Exercise 9.8 - Scattering by a slightly lossy dielectric sphere (Jackson 10.4)

An unpolarized wave of frequency $\omega = ck$ is scattered by *slightly* lossy uniform isotropic dielectric sphere of radius R much smaller than a wavelength. The sphere is characterized by an ordinary real dielectric constant ϵ_r, and a real conductivity σ. The parameters are such that the skin depth δ is very *large* compared to the radius R.

(a) Calculate the differential and total *scattering* cross section.

(b) Show that the absorption cross section is

$$\sigma_{\text{abs}} = 12\pi R^2 \frac{R Z_0 \sigma}{(\epsilon_r + 2)^2 + (Z_0\sigma/k)^2}$$

Exercise 9.9 - Radiation of line and sheet sources (Jackson 6.1)

In three dimensions the solution to the wave equation

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}_t \ ,$$

where t notes the *transverse* current, for a point source in space and time (a light flash at $t' = 0, \mathbf{x}' = 0$) is a spherical shell distrubance of radius $R = ct$, namely the Green function $G^{(+)}$:

$$G^{(\pm)}(\mathbf{x}, t; \mathbf{x}', t') = \frac{\delta\left(t' - \left[t \pm \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right]\right)}{|\mathbf{x} - \mathbf{x}'|} \ .$$

It may be initially surprising that in one or two dimensions, the disturbance possesses a "wake", even though the source is a "point" in space and time. The solutions for fewer dimenssions than three can be found by superposition in the superfluous dimension(s), to eliminate dependence on such variable(s). For example, a flashing line source of uniform amplitude is equivalent to a point source in two dimensions.

(a) Starting from the retarded solution to the three-dimensional wave equation:

$$\Psi(\mathbf{x}, t) = \int \mathrm{d}^3 x \frac{[f(\mathbf{x}', t')]_{\text{ret}}}{|\mathbf{x} - \mathbf{x}'|}$$

with the retarded time $t' = t - |\mathbf{x} - \mathbf{x}'|/c$, show that the source $f(\mathbf{x}', t') = \delta(x')\delta(y')\delta(t')$, equivalent to a $t = 0$ point source at the origin in two dimensions, produces a two-dimensional wave,

$$\Psi(x, y, t) = \frac{2c\Theta(ct - \varrho)}{\sqrt{c^2 t^2 - \varrho^2}} \ ,$$

where $\varrho = x^2 + y^2$ and $\Theta(\xi)$ is the unit step function.

(b) Show that a "sheet" source, equivalent to a point pulsed source at the origin in one space dimension, produces a one-dimensional wave proportional to

$$\Psi(x, t) = 2\pi c\Theta(ct - |x|) \ .$$

Exercise 9.10 - Dipole and quadrupole radiation in real time (Jackson 9.7)

(a) By means of Fourier superposition of different frequencies or equivalent means, show for a real electric dipole $\mathbf{p}(t)$ that the isntantaneous radiated power per unit solid angle at a distance r from the dipole in a direction $\mathbf{n}$ is

$$\frac{\mathrm{d}P(t)}{\mathrm{d}\Omega} = \frac{Z_0}{16\pi^2 c^2}\left|\left[\mathbf{n} \times \frac{\mathrm{d}^2\mathbf{p}}{\mathrm{d}t'^{\,2}}(t')\right] \times \mathbf{n}\right|^2$$

where $t' = t - r/c$ is the retarded time. For a magnetic dipole $\mathbf{m}(t)$, substitute $(1/c)\ddot{\mathbf{m}} \times \mathbf{n}$ for $(\mathbf{n} \times \ddot{\mathbf{p}}) \times \mathbf{n}$.

(b) Show similarly for a real quadrupole tensor $Q_{\alpha\beta}(t)$given by

$$Q_{\alpha\beta} = \int \mathrm{d}^3x(3x_\alpha x_\beta - r^2\delta_{\alpha\beta})\rho(\mathbf{x})$$

with a real charge density $\rho(x,t)$ that the instantaneous radiated power per unit solid angle is

$$\frac{\mathrm{d}P(t)}{\mathrm{d}\Omega} = \frac{Z_0}{576\pi^2c^4}\left|\left[\mathbf{n} \times \frac{\mathrm{d}^3\mathbf{Q}}{\mathrm{d}t'^{\,3}}(\mathbf{n},t')\right] \times \mathbf{n}\right|^2$$

where $Q_\alpha(\mathbf{n},t') = \sum_\beta Q_{\alpha\beta}$.

Exercise 9.11 - Corrections to dipole approximation (Jackson 10.5)

The scattering by the dielectric sphere Exercise 9.8 was trated as purely electric dipole scattering. This is adequate unless it happens that the real dielectric constant ϵ/ϵ_0 is very large. In these circumstances a magnetic dipole contribution, even though higher order in kR, may be important.

(a) Show that the changing magnetic flux of the incident wave induces an azimuthal current flow in the sphere and produces a magnetic dipole moment

$$\mathbf{m} = \frac{\imath 4\pi\sigma Z_0}{k\mu_0}(kR)^2\frac{R^3}{30}\mathbf{B}_{\text{inc}}$$

(b) Show that application of the optical theorem to the coherent sum of the electric and magnetic dipole contribution leads to a total cross section

$$\sigma_t = 12\pi R^2(RZ_0\sigma)\left[\frac{1}{(\epsilon_r+2)^2 + (Z_0\sigma/k)^2} + \frac{1}{90}(kR)^2\right]$$

(Compare: Landau Lifshitz, *Electrodynamics of Continuous Media*, p.323)