Class 3 - Laplace equation in cylindrical and polar coordinates

Class material

Exercise 3.1 - Cylinder with potential on the cover

Consider a metal cylinder with radius R and height h. The top cover of the cylinder is maintained at a fixed potential of V_0, while the cylindrical sheath and the bottom cover is held at $V = 0$. What is the electric potential inside the cylinder?

(a) Write the Laplace equation and the boundary conditions for the potential!

(b) Write the Laplace equation in cylindrical coordinates for the case of rotation symmetry around the z axis!

(c) Determine the expansion coefficients from the boundary conditions!

Exercise 3.2 - Cylinder with alternating potential on its sheath

On the sheath of an infinite cylinder we maintain a potential as shown in Fig.1. Determine the potential in the entire space!

(a) Write the Laplace equation and the corresponding boundary conditions!

(b) Give the general solution in cylindrical coordinates using translational symmetry in the z-direction!

(c) Determine the expansion coefficients from the boundary conditions!

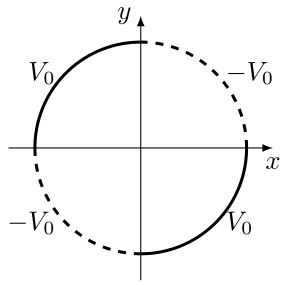

Figure 1

Exercise 3.3 - Semi-infinite cylinder with fixed potential on the bottom

Given a semi-infinite metal cylinder wall along the $+z$ axis with radius a. We close the bottom of the cylinder with a metal circle, which we insulate from the walls and we maintain it at a potential of V_0. The walls are grounded.

(a) Give the $\Phi(\vec{r})$ electrostatic potential inside the cylinder!

(b) Calculate the $\vec{E}$ electric field inside the cylinder at the centre of the base circle (at the $z = 0+$)!

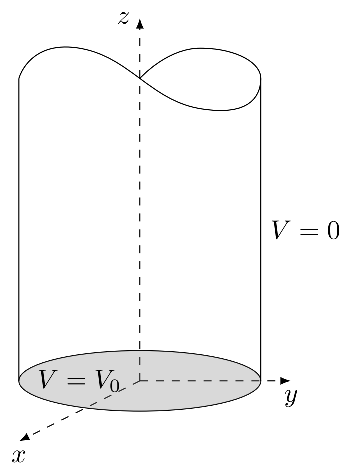

Figure 2

Exercise 3.4 - Hemisphere at different potentials with a grounded sphere inside

We take a spherical shell with radius b and we cut it in half and insulate the two resulting semi-spheres. We then put the upper(lower) hemisphere at potential $+V_0(-V_0)$. Inside the sphere we put an other metal sphere with radius $a < b$ and we ground it (V=0). What is the $\Phi(\vec{r})$ potential function between the two spheres?

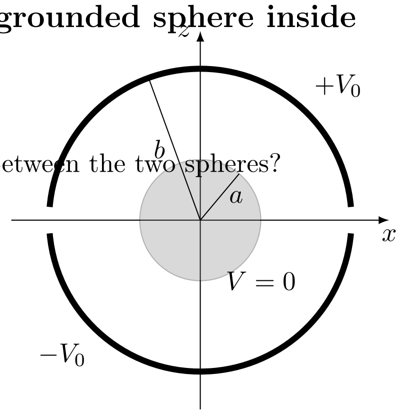

Figure 3

Homework

The following problems (marked with an asterisk) form the basis of the short test at the beginning of the next class.

Exercise 3.5 - Infinite cylinder*

Given an infinite metal cylinder along the $+z$ axis with radius a. We cut the cylinder at the $z = 0$ plane and insulate the two parts from each other and we place the upper part of the cylinder ($z > 0$) at a potential of $+V_0$, and the lower part at $-V_0$.

(a) Calculate $\Phi(\vec{r})$, the electrostatic potential inside the cylinder!

(b) Calculate the $\vec{E}$ electric field at the origin up to first order!

Hint:

$$\frac{2}{\pi}\int_0^\infty \frac{\sin kz}{k}\mathrm{d}k = \begin{cases} +1 & \text{if} \quad z > 0 \\ -1 & \text{if} \quad z < 0 \end{cases}$$

Exercise 3.6 - Finite cylinder*

Consider the previous exercise in the case, when we the cylinder is finite, extending between $\pm b$. We achieve this by cutting the cylinder of the previous problem at $\pm b$ and closing the cylinder at both ends using two metallic disks welded to the original cylinder. As a result the disks will also be at a potential of $\pm V_0$.

(a) Calculate the $\Phi(\vec{r})$ electrostatic potential inside the cylinder!

(b) Calculate the $\vec{E}$ electric field along the z axis inside the cylinder!

Exercise 3.7 - Circular segment with potential on the arc*

Consider a 2 dimensionsal circular segment with subtense ϕ_0. We fix the potential on $V = 0$ on the bounding radii and on $V(\phi) = V_0$ on the arc as shown on Fig.5. What are the electric potential and the electric field inside the circular segment?

(a) Write the Laplace equation and the corresponding boundary conditions!

(b) Give the general solution in 2 dimensional polar coordinates!

(c) Determine the expansion coefficients from the boundary conditions!

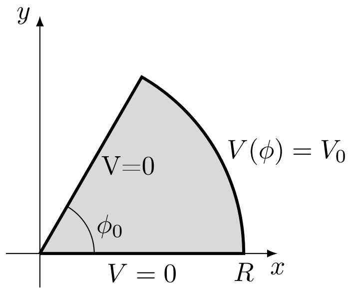

Figure 5

These problems are for further practice and to have some fun!

Exercise 3.8 - Integral form of exact solution in polar coordinates (Jackson 2.12)

Starting with the series solution

$$\Phi(\rho,\phi) = a_0 + b_0 \ln \rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\phi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\phi + \beta_n)$$

for two-dimensional potential problem with the potential specified on the surface of the cylinder of radius b, evaluate the coefficients formally, substitute them into the series, and sum it to obtain the potential *inside* the cylinder in the form of Poisson's intergal:

$$\Phi(\rho,\phi) = \frac{1}{2\pi}\int_0^{2\pi} \Phi(b,\phi') \frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\phi'-\phi)} \mathrm{d}\phi'$$

What modification is necessary if the potential is desired in the region of space bounded by the cylinder and infinity?

Exercise 3.9 - Infinitely long cylinder cut in two halves along length (Jackson 2.13)

(a) Two halves of a lond hollow conducting cylinder of inner radius b are separated by small lengthwise gaps on each side, and are kept at different potentials V_1 and V_2. Show that the potential inside is given by

$$\Phi(\rho,\phi) = \frac{V_1+V_2}{2} + \frac{V_1-V_2}{\pi}\tan^{-1}\left(\frac{2b\rho}{b^2-\rho^2}\cos\phi\right)$$

where ϕ is measured from a plane perpendicular to the plane through the gap.

(b) Calculate the surface-charge density on each half of the cylinder.

Exercise 3.10 - Infinitely long cylinder with four segments along length at alternating potentials(Jackson 2.14)

A variant of the preceding two-dimensional problem is a long hollow conducting cylinder of radius b that is divided into equal quarters, alternate segments being held at potential $+V$ and $-V$.

(a) Solve by means of the series solutions and show that the potential inside the cylinder is

$$\Phi(\rho,\phi) = \frac{4V}{\pi}\sum_{n=a}^{\infty}\left(\frac{\rho}{b}\right)^{4n+2}\frac{\sin[(4n+2)\phi]}{2n+1}$$

(b) Sum the series and show that

$$\Phi(\rho,\phi) = \frac{2V}{\pi}\tan^{-1}\left(\frac{2\rho^2 b^2 \sin 2\phi}{b^4-\rho^4}\right)$$

(c) Sketch the field lines and equipotentials.

Exercise 3.11 - Finite cylinder cut in half along length (Jackson 3.9+3.10)

A hollow right circular cylinder of radius b has its axius coincident with the z axis and its ends at $z=0$ and $z=L$. The potential on the end faces is zero, while the potential on the cylindrical surface is given as $V(\phi, z)$.

(a) Using the appropriate separation of variables in cylindrical coordinates, find the series solution for the potential anywhere inside the cylinder. Consider the following potential:

$$V(\theta, z) = \begin{cases} V & \text{if} \quad -\pi/2 < \phi < \pi/2 \\ -V & \text{if} \quad \pi/2 < \phi < 3\pi/2 \end{cases}$$

(b) Find the potential inside the cylinder.

(c) Assuming $L \gg b$, consider the potential at $z = L/2$ as a function of ρ and ϕ polar coordinates and compare it with the two-dimensional problem in Exercise 3.9.