Class 2 - Laplace equation in spherical coordinate systems

Class material

Exercise 2.1 - Sphere with azimuthal symmetry

Consider a sphere with radius R, for which on the surface of the upper semi-sphere the potential is held at V_0, while the lower is held at $-V_0$. Compute the electric potential and the electric field in the whole space.

(a) Write down the Laplace equation in spherical coordinates using the asimuthal symmetry and the boundary conditions for the potential.

(b) Determine the expansion coefficients from the boundary conditions.

(c) Determine the electric field from the potential.

Exercise 2.2 - Circular disc held at a fixed potential (Jackson 3.3)

A thin, flat, conducting, circular disc of radius R is located in the $x-y$ plane with its center at the origin and is maintained at fixed potential V. With the information that the charge density on a disc at fixed potential is proportional to $(R^2-\rho^2)^{-1/2}$, where ρ is the distance out from the center of the disc.

(a) Show that for $r > R$ the potential is

$$\Phi(r,\theta,\phi) = \frac{2V}{\pi}\frac{R}{r}\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{2\ell+1}\left(\frac{R}{r}\right)^{2\ell} P_{2\ell}(\cos\theta)$$

(b) Find the potential for $r < R$.

(c) What is the capacitance of the disc?

Exercise 2.3 - Sphere with wedges kept at alternating potential (Jackson 3.4)

The surface of a hollow conducting sphere of inner radius a is divided into an *even number* of equal segments by a set of planes; there common line of intersection is the z axis and they are distributed uniformly in the angle ϕ. (The segments are like the skin on wedges of an apple, or the earth's surface between successice meridians of longitude.) the segments are kept at fixed potentials $\pm V$,alternately.

(a) Set up a series representation for the potential inside the sphere for the general case of $2n$ segments, and carry the calculation of the coefficients in the series far enough to determine exactly which coefficients are different from zero. For the nonvanishing terms, exhibit the coefficients as an integral over $\cos\theta$.

(b) For the special case of $n = 1$ (two hemispheres) determine explicitly the potential up to and including all terms with $\ell = 3$. By a coordinate transformation verify that this reduces to

$$\Phi(r,\theta) = V\left[\frac{3}{2}\frac{r}{a}P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{a}\right)^5 P_5(\cos\theta)\dots\right]$$

where we cut the sphere horizontally.

Exercise 2.4 - Boundary value problems with azimuthal symmetry (Jackson, chapter 3.3)

If the potential of an azimuthally symmetric system is known along the z axis, and is of the form,

$$\Phi(z=r) = \sum_{l=0}^{\infty} \left[A_l r^l + \frac{B_l}{r^{l+1}} \right], \tag{1}$$

it is possible to extend it to all space by simply multiplying each term by the corresponding Legendre polynomial,

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\theta). \tag{2}$$

Make use of this idea to solve the following problems!

1. Show that the potential due to a unit point charge at position $\mathbf{x}'$ at point $\mathbf{x}$ may be written as

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma). \tag{3}$$

   where γ denotes the angle between $\mathbf{x}$ and $\mathbf{x}'$. To show this, first assume that the charge is situated on the z-axis, and place also the point $\mathbf{x}$ on the z-axis too. The resulting potential in this case is,

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{|r-r'|}. \tag{4}$$

   Expand Eq. (4) and apply Eqs. (1) and (2) to show that Eq. (3) holds.

2. Consider a ring with radius a and with total charge q distributed uniformly. The ring is situated in the $z=b$ with its center at the point $(0,0,b)$. Show that its potential can be written as,

$$\Phi(r,\theta) = \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\alpha) P_l(\cos\theta), \tag{5}$$

   where $r_<(r_>)$ is the smaller(larger) of r and $c=\sqrt{a^2+b^2}$, and $\cos\alpha = b/c$.

Homework

The following problems (marked with an asterisk) are for homework.

Exercise 2.5 - Charged sphere with azimuthal symmetry*

Consider a sphere with radius R and surface charge density $\sigma = \sigma_0 \cos\theta$. Compute the electric potential and the electric field in the whole space.

(a) Write down the Laplace equation in spherical coordinates using the asimuthal symmetry and the boundary conditions for the potential.

(b) Determine the expansion coefficients from the boundary conditions.

(c) Determine the electric field from the potential.

Exercise 2.6 - Two concentric spheres (Jackson 3.1)*

Two concentric spheres have radii a, b ($a < b$) and each is divided into two hemispheres by the same horizontal plane. The upper hemisphere of the inner sphere and the lower hemisphere of the outer sphere are maintained at potential V. The other hemispheres are at zero potential. Determine the potential in the region $a \leq r \leq b$ as a series in Legendre polynomials. Include terms at least up to $\ell = 4$. Check your solution against known results in the limiting cases $b \to \infty$, and $a \to 0$.

Exercise 2.7 - Sphere with a neutral cap (Jackson 3.2)*

A spherical surface of radius R has charge uniformly distributed over its surface with density $Q/4\pi R^2$, except for a spherical cap at the north pole, defined by the cone $\theta = \alpha$. p

(a) Show that the potential inside the spherical surface can be expressed as

$$\Phi = \frac{Q}{8\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{1}{2\ell+1} \left[P_{\ell+1}(\cos\alpha) - P_{\ell-1}(\cos\alpha)\right] \frac{r^\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

where, for $\ell = 0$, $P_{\ell-1}(\cos\alpha) = -1$. What is the potential outside?

(b) Find the electric field vector at the origin.

(c) Discuss the limiting forms of the potential from part (a) and the electric field from part (b) as the spherical cap becomes (1) very small, and (2) so large that the area with charge on it becomes a very small cap at the south pole.

These problems are for further practice and to have some fun!

Exercise 2.8 - Oppositely charged conducting hemispheres (Jackson 2.22)

(a) Show that, for oppositely charged conducting hemispherical shells separated by a tiny gap, the interior potential ($r < a$) in the z axis is

$$\Phi_{\text{in}}(z) = V\frac{a}{z}\left[1 - \frac{a^2 - z^2}{a\sqrt{a^2 + z^2}}\right]$$

Find the first few terms of the expansion in powers of z and show that they agree with

$$\Phi(x, \theta, \phi) = \frac{3Va^2}{2x^2}\left[\cos\theta - \frac{7a^2}{12x^2}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right)\ldots\right]$$

with the appropriate substitutions.

(b) From the result of part (a) show that the radial electric field on the positive z axis is

$$E_r(z) = \frac{Va^2}{(z^2+a^2)^{3/2}}\left(3+\frac{a^2}{z^2}\right)$$

for $z > a$, and

$$E_r(z) = -\frac{V}{a}\left[\frac{3+(a/z)^2}{(1+(z/a)^2)^{3/2}} - \frac{a^2}{z^2}\right]$$

for $|z| < a$. Show that the second form is well behaved at the origin, with the value, $E_r(a) = -3V/2a$. Show that at $z = a$ (north pole inside) it has the value $-(\sqrt{2}-1)V/a$. Show that the radial field at the north pole outside has the value $\sqrt{2}V/a$.

(c) Make a sketch of the electric field lines, both inside and outside the conducting hemispheres, with directions indicated. Make a *plot* of the radial electric field along the z axis from $z = -2a$ to $z = 2a$.

Exercise 2.9 - Line charge inside a conducting sphere (Jackson 3.14)

A line charge of length $2d$ with a total charge Q has a linear charge density varying as $(d^2 - z^2)$, where z is the distance from the midpoint. A grounded, conducting, spherical shell of inner radius $b > d$ is centered at the midpoint of the line charge.

(a) Find the potential everywhere inside the spherical shell as an expansion in Legendre polynomials.

(b) Calculate the surface-charge density induced on the shell.

(c) Discuss your answers to parts (a) and (b) in the limit that $d \ll b$.

Exercise 2.10 - Simplified model of a battery (Jackson 3.15)

Consider the following 'spherical cow' model of a battery connected to an external circuit. A sphere of radius a and conductivity σ is embedded in a uniform medium of conductivity σ'. Inside the sphere there is a uniform (chemical) force in the z direction acting on the charge carriers; its strength as an effective electric field entering Ohm's law is F. In the steady state, electric fields exist inside and outside the sphere and surface charge resides on its surface.

(a) Find the electric field (in addition to F) and current density everywhere in space. Determine the surface-charge density and show that the electric dipole moment of the spheres is $\rho = 4\pi\epsilon_0\sigma a^3 F/(\sigma + 2\sigma')$.

(b) Show that the total current flowing out through the upper hemisphere of the sphere is

$$I = \frac{2\sigma\sigma'}{\sigma + 2\sigma'} \cdot \pi a^2 F$$

Calculate the total power dissipated outside the sphere. Using the lumped circuit relations, $P = I^2 R_e = IV_e$, find the effective external resistance R_e and voltage V_e.

(c) Find the power dissipated within the spheres and deduce the effective internal resistance R_i and voltage V_i.

(d) Define the total voltage through the relation $V_t = (R_e + R_i)I$ and show that $V_t = 4aF/3$, as well as $V_e + V_i = V_t$. Show that IV_t is the power supplied ny the 'chemical' force.

Reference: W.M. Saslow, *Am. J. Phys.* **62**, 495-501 (1994).