Class 1 - Laplace equation in Cartesian coordinate systems

Class material

Exercise 1.1 - Semi-infinite rectangle

On the boundary of a two-dimensional semi-infinite rectangle the potential is fixed as shown on Fig.1. Compute the electric potential and the electric field inside the rectangle.

(a) Write down the Laplace equation for the potential in two dimensions, and specify the corresponding boundary conditions.

(b) Using the separation of variables in Cartesian coordinates, reduce the Laplace equation to ordinary differential equations.

(c) Solve the equations and impose the boundary conditions.

(d) Determine the electric field from the potential.

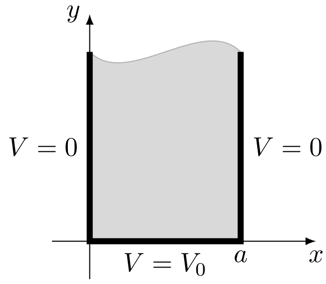

Figure 1

Exercise 1.2 - Infinite planes with alternating potential

Consider two parallel conducting planes at distance h, positioned at $z = 0$ and $z = h$ respectively. The one at $z = h$ is on zero potential, while the one at $z = 0$ is sliced into stripes of width b parallel to the x axis, which are insulated from each other, and then impose an alternating potential $\pm V_0$ as illustrated in Fig.**??**.

(a) Compute the electrostatic potential $\Phi(x, y, z)$ between the planes.

(b) Determine the $\vec{E}$ eletric field along the z axis in the $0 \leq z \leq h$ range.

Exercise 1.3 - Square tube-I

Consider an infinitely long metal tube along direction z of square cross section with sides of length a. The side walls are insulated from each other along the edges. We connect the opposite sides to each other; one pair is grounded, while the other is kept at a potential $V = V_0$ as shown in Fig.3.

(a) Write the Laplace equation for the electrostatic field inside the tube in a suitable coordinate system.

(b) Specify the corresponding boundary conditions, and write down the required orthogonality relations.

(c) Determine the electrostatic potential inside the tube.

(d) Determine the electrostatic field inside the tube.

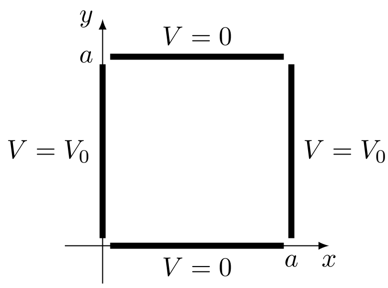

Figure 3

Exercise 1.4 - Green's function for a square (Jackson 2.15)

(a) Show that the Green's function $G(x, y; x', y')$ appropriate for Dirichlet boundary conditions for a square two-dimensional region, $0 <= x <= 1$, $0 <= y <= 1$, has an expansion

$$G(x, y; x', y') = 2 \sum_{n=1}^{\infty} g_n(y, y') \sin(n\pi x) \sin(n\pi x')$$

where $g_n(y, y')$ satisfies

$$\left(\frac{\partial^2}{\partial y'^2} - n^2\pi^2 \right) g_n(y, y') = -4\pi\delta(y' - y) \qquad \text{and} \qquad g_n(y, 0) = g_n(y, 1) = 0$$

(b) Taking for $g_n(y, y')$ appropriate linear combinations of $\sinh(n\pi y')$ and $\cosh(n\pi y')$ in the two regions $y' < y$ and $y' > y$, in accord with the boundary conditions and the discontinuity in slope required by the source delta function, show that the explicit form of G is

$$G(x, y; x', y') = 8 \sum_{n=1}^{\infty} \frac{1}{n \sinh(n\pi)} \sin(n\pi x) \sin(n\pi x') \sinh\big(n\pi y_{\langle}\big) \sinh\big(n\pi(1 - y_{\rangle})\big)$$

where $y_{\langle}$ ($y_{\rangle}$) is the smaller (larger) of y and y'.

Homework

The following problems (marked with an asterisk) form the basis of the short test at the beginning of the next class.

Exercise 1.5 - Infinite planes*

Consider two infinite charged metallic planes parallel to the xy plane, one at $z = 0$ and the other at $z = 0.4$. The potential on the planes in some arbitrary units is:

$$\Phi(x, y, 0) = 5\sin(4x)\cos(3y)$$
$$\Phi(x, y, 0.4) = 2\sin(4x)\cos(3y)$$

(a) Write the Laplace equation for the electrostatic field between the two planes in a suitable coordinate system, and specify the corresponding boundary conditions.

(b) Determine the electric potential $\Phi(x, y, z)$ between the planes.

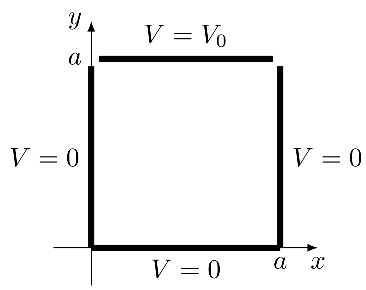

Figure 4

Exercise 1.6 - Square tube-II*

Consider an infinitely long metal tube along direction z of square cross section with sides of length a. One of the side walls given by $y = a$ equation is insulated from the other sides and is held at a potential V_0 potential, while 'U' formed by the other three sides is grounded as shown on Fig.4.

(a) Write the Laplace equation for the electrostatic field inside the tube in a suitable coordinate system.

(b) Specify the corresponding boundary conditions, and write down the required orthogonality relations.

(c) Determine the electrostatic potential inside the tube.

(d) Determine the electrostatic field inside the tube. What is the value at the middle?

Exercise 1.7 - Charged infinite plane*

The surface electric charge density on an infinite plane lying at $z = 0$ is:

$$\sigma(x, y) = \sigma_0 \sin(\alpha x)\sin(\beta y)$$

(a) Write the Laplace equation in a suitable coordinate system for the region $z > 0$, and specify the corresponding boundary conditions.

(b) Determine the electric potential $\Phi(x, y, z)$ in the region $z > 0$.

These problems are for further practice and to have some fun!

Exercise 1.8 - Potential of a square from the Green's function (Jackson 2.16)

A two dimensional potential exists on a unit square area ($0 \leq x \leq 1$, $0 \leq y \leq 1$) bounded by 'surfaces' held at zero potential. Over the entire square there is a uniform charge density of unit strength (per unit length in z). Using the Green's function of Exercise 1.4, show that the solution can be written as

$$\Phi(x,y) = \frac{4}{\pi^2 \epsilon_0} \sum_{m=0}^{\infty} \frac{\sin[(2m+1)\pi x]}{(2m+1)^3} \left\{ 1 - \frac{\cosh[(2m+1)\pi(y-1/2)]}{\cosh[(2m+1)\pi/2]} \right\}$$