Dirac delta, Green függvény

1. Feladat
   Határozzuk meg a következő integrálokat:

   (a)
   $$\int_{-3}^{3} \sin(x)\delta(x-\frac{\pi}{6})dx =$$

   (b)
   $$\int_{-3}^{3} \mathrm{tg}(x)\delta(x-\pi)dx =$$

   (c)
   $$\int_{-3}^{3} x^3\delta(x^2+x-12)dx =$$

   (d)
   $$\int_{0}^{\infty} e^{-x}\delta(\sin(x))dx =$$

2. A Dowkins köny mellékelt példája esetén határozzuk meg differenciálegyenletek Green függvényét és a partikuláris megoldást!

$$\dot{y} + p(t)y = q(t), \qquad q(a) = 0$$
$$G(t,t') = e^{-\int_{t'}^{t} p(x)dx}\Theta(t-t'), \qquad t > a$$

3. Határozzuk meg a Green függvény segítségével a következő egyenlet partikuláris megoldását:

$$\dot{y} + \mathrm{tg}(t)y = \sin(t), \qquad t > 0$$

Example 4 **Find the solution to the following IVP.**

$$t\,y' + 2y = t^2 - t + 1 \qquad\qquad y(1) = \frac{1}{2}$$

Solution

First, divide through by the *t* to get the differential equation into the correct form.

$$y' + \frac{2}{t}y = t - 1 + \frac{1}{t}$$

Now let's get the integrating factor, $\mu(t)$.

$$\mu(t) = \mathbf{e}^{\int \frac{2}{t}dt} = \mathbf{e}^{2\ln|t|}$$

Now, we need to simplify $\mu(t)$. However, we can't use (11) yet as that requires a coefficient of one in front of the logarithm. So, recall that

$$\ln x^r = r \ln x$$

and rewrite the integrating factor in a form that will allow us to simplify it.

$$\mu(t) = \mathbf{e}^{2\ln|t|} = \mathbf{e}^{\ln|t|^2} = |t|^2 = t^2$$

We were able to drop the absolute value bars here because we were squaring the *t*, but often they can't be dropped so be careful with them and don't drop them unless you know that you can. Often the absolute value bars must remain.

Now, multiply the rewritten differential equation (remember we can't use the original differential equation here…) by the integrating factor.

$$\left(t^2 y\right)' = t^3 - t^2 + t$$

Integrate both sides and solve for the solution.

$$\begin{aligned} t^2 y &= \int t^3 - t^2 + t\,dt \\ &= \frac{1}{4}t^4 - \frac{1}{3}t^3 + \frac{1}{2}t^2 + c \\ y(t) &= \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{c}{t^2} \end{aligned}$$

Finally, apply the initial condition to get the value of *c*.

$$\frac{1}{2} = y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + c \qquad \Rightarrow \qquad c = \frac{1}{12}$$

The solution is then,

$$y(t) = \frac{1}{4}t^2 - \frac{1}{3}t + \frac{1}{2} + \frac{1}{12t^2}$$

Here is a plot of the solution.