Bell's inequality and quantum correlations

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The conflict between quantum predictions and some "common sense" assumptions is explored at a level suitable for undergraduates with an elementary knowledge of quantum mechanics. The current status of experimental tests of the conflict is reviewed, and the implications are briefly discussed.

I. INTRODUCTION

Bell's theorem[1] has still not penetrated the standard undergraduate physics major curriculum, as a brief survey of standard undergraduate quantum mechanics texts will show. Although the theorem is less than 20 years old, which is fairly new for "textbook" treatment, this absence is probably also partly because previously the theorem has been hard to grasp, leading Peres to remark, for example, that "as it often happens, the subtlety of nature beggars the imagination."[2]

However, recent generalizations of the proof by d'Espagnat[3] and Stapp[4] offer the possibility of displaying the conflict between quantum theory and "common sense" assumptions in a much simpler fashion. The intention of this paper is to explicitly do just that. The level is appropriate for undergraduates with an elementary knowledge of quantum mechanics, such as the first six chapters of Feynman, Vol. III, or the equivalent.[5]

In Sec. II of this paper, we derive the quantum mechanics of a correlation experiment. Sections III and IV then derive the Bell inequality for this same experiment in two different ways, and show the inconsistency with quantum mechanics. Section V reviews the experimental tests and discusses the assumptions used in deriving the Bell inequality.

II. QUANTUM PREDICTION

The original realization of a conflict between quantum theory and "reasonable" assumptions was by Einstein, Podolsky, and Rosen[6] in 1935 (EPR). EPR concluded that "the description of reality as given by a wave function is not complete." The assumptions of what constitutes "complete" for EPR are fairly complex; recently McGrath has cast the EPR paper into symbolic logic[7] and clearly displays the features of their argument.

Bohm[8] called the EPR argument a "paradox," and we shall follow Bohm's analysis of the quantum predictions here. We imagine a system consisting of a molecule containing two identical atoms. The total spin of the molecule is zero, and the spin of each atom is $\hbar/2$. The molecule is split by a process that does not change the total angular momentum, and each atom travels in opposite directions. Each atom then has the z component of its spin measured by a Stern–Gerlach[9] apparatus, as in Fig. 1(a). The source S sends the two atoms, 1 and 2, to the two vertically oriented Stern–Gerlach magnets P_1 and P_2. (The P stands for polarization.) It will be important to notice the arrows painted on P_1 and P_2 which indicate what direction is "up" for each individual analyzer.

Atom 1 has a wave function $U(1)$, which is an eigenvector of the operator P_1. If atom 1 is spin up, we label the wave function $U_+(1)$, and we adopt the notation that the eigenvalue is $+1$:

$$P_1 U_+(1) = + U_+(1) . \tag{1}$$

If atom 1 is spin down, we label the wave function as $U_-(1)$, and

$$P_1 U_-(1) = - U_-(1) . \tag{2}$$

The wave functions for atom 2 we similarly write as $U(2)$, with similar eigenvalues:

$$P_2 U_+(2) = + U_+(2) ,$$
$$P_2 U_-(2) = - U_-(2) . \tag{3}$$

The total wave function for the system under consideration is

$$\Psi = (1/\sqrt{2})[U_+(1)U_-(2) - U_-(1)U_+(2)] . \tag{4}$$

That this wave function is correct is shown in the Appendix.

Next we extend Bohm's analysis and consider the situation shown in Fig. 1(b). The two Stern–Gerlach analyzers, P'_1 and P'_2, have been rotated about the y axis by angles θ_1 and θ_2, respectively. We know that each atom has a definite spin component in any direction specified by the orientation of the polarizers, so we may write down the eigenvectors U' for these orientations:

$$P'U'_\pm = \pm U'_\pm . \tag{5}$$

The total wave function will not be just Eq. (4) with primes added unless $\theta_1 = \theta_2$. This is because the two apparatuses will not define the same directions as up.

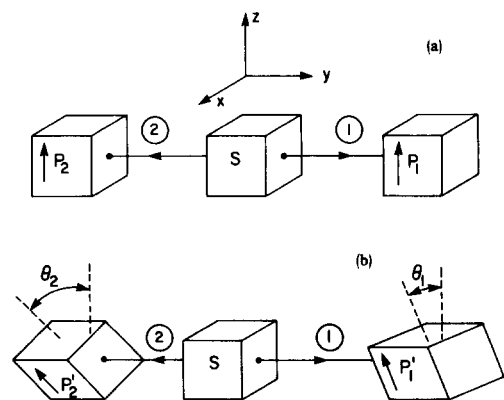In fact, to determine the total wave function, we use the

Fig. 1. Measuring the spin of the two atoms. (a) The orientation of the two polarizers is the same, and measures the z component of the spin. (b) Each polarizer P'_i has been rotated about the y axis by an angle θ_i.

standard rotation matrix applied to each eigenfunction individually:

$$U_+(1) = \cos(\theta_1/2)U'_+ (1) - \sin(\theta_1/2)U'_- (1),$$

$$\tag{6}$$

$$U_-(1) = \cos(\theta_1/2)U'_- (1) + \sin(\theta_1/2)U'_+ (1),$$

and similarly for 2. The reason for the appearance of $\theta_1/2$ instead of θ_1 is related to the fact that we have spin $\frac{1}{2}$ objects and know, for example, that if $\theta_1 = 180°$, $U_-(1) = U'_+ (1)$. The sign convention is due to the fact we are writing the unrotated eigenvectors in terms of the rotated eigenvectors, not vice versa. Inserting Eq. (6) and its equivalent for atom 2 into the total wave function, Eq. (4), and collecting terms gives

$$\Psi = \frac{1}{\sqrt{2}}\left[\cos\left(\frac{\theta_1-\theta_2}{2}\right)U'_+ (1)U'_- (2)\right.$$
$$- \cos\left(\frac{\theta_1-\theta_2}{2}\right)U'_- (1)U'_+ (2)$$
$$- \sin\left(\frac{\theta_1-\theta_2}{2}\right)U'_+ (1)U'_+ (2)$$
$$\left.- \sin\left(\frac{\theta_1-\theta_2}{2}\right)U'_- (1)U'_- (2)\right].$$

$$\tag{7}$$

Note that for $\theta_1 = \theta_2$, Eq. (7) is identical to Eq. (4) but with primes added.

In Secs. III and IV we shall display two different proofs of Bell's theorem, for which we will require two different pieces of information from Eq. (7). First, we will ask what is the probability that P'_1 measures atom 1 to be spin up simultaneous to P'_2 measuring the companion atom 2 to also be spin up. The answer is just the square of the coefficient in front of the $U'_+ (1)U'_+ (2)$ term of Eq. (7):

$$\text{prob(up,up)} = \tfrac{1}{2} \sin^2[(\theta_1 - \theta_2)/2].$$

$$\tag{8}$$

Note that for $\theta_1 = \theta_2$, this probability is zero as expected.

The other piece of information we shall require is the expectation value of the product of the eigenvalues. Say P'_1 gives a signal $p'_1 = \pm 1$ depending on whether atom 1 is spin up or spin down, and P'_2 gives a similar signal $p'_2 = \pm 1$ for atom 2. We multiply these two values together to form $p'_1 p'_2$. Then the expectation value for the product of the two signals, which would be the average value one would expect after a large number of trials, is

$$\Psi^*P'_1 P'_2 \Psi = - \cos(\theta_1 - \theta_2),$$

$$\tag{9}$$

which can be shown directly from Eq. (7).

III. D'ESPAGNAT'S PROOF OF THE BELL INEQUALITY

As discussed above, EPR made a large number of assumptions about what constitutes a complete theory, and showed that the predictions of quantum mechanics violate those assumptions. The Bell inequality can be viewed as a generalization of EPR in which fewer but more general
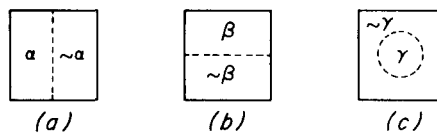

Fig. 2. Representing the number of objects with property α vs $\sim\alpha$, β vs $\sim\beta$, and γ vs $\sim\gamma$.

assumptions are made. Bell's original proof[1] was in the context of hidden variable theories, but since then it has been generalized even further. Here we shall prove the Bell inequality two different ways. In this section we present a simplified version of the proof of d'Espagnat[3]; in Sec. IV we discuss a proof due to Stapp.[4] It should be stressed at the outset that the results of both of these will conflict with the results of Sec. II and with experimental tests. Thus the assumptions being made here are at least partly incorrect. Further, our experience indicates that the greater one's familiarity with quantum mechanics the less reasonable these assumptions will appear.

Bell himself has applauded d'Espagnat's generalization of the proof.[10] Unfortunately, many physicists and students, including myself, find the number of steps in this proof sufficiently large that it is difficult to see the main features clearly. Thus here we present a simplified version.

Our proof begins by discussing a classical situation which does not generalize to the quantum theory of spin measurements. We consider a collection of macroscopic physical objects, each characterized by three independent two-valued parameters α, β, γ. For example, the objects can be either wood or metal, either black or white, and either a sphere or a cube, as shown in Table I. Following d'Espagnat, we represent the number of objects with given properties as Venn diagrams as in Fig. 2. Then Fig. 3 shows the number of objects N with properties $(\alpha,\sim\beta)$, $(\beta,\sim\gamma)$, and $(\alpha,\sim\gamma)$, respectively. Inspection shows that

$$N(\alpha,\sim\beta) + N(\beta,\sim\gamma) \geqslant N(\alpha,\sim\gamma)$$

$$\tag{10}$$

with the equality holding only if there are no objects in certain categories.[11] Also, it is important to note that Eq. (10) is based on an incomplete labelling of the objects in the sample. For example, the first term of Eq. (10), $N(\alpha,\sim\beta)$, includes two categories of objects: $N(\alpha,\sim\beta,\gamma)$ and $N(\alpha,\sim\beta,\sim\gamma)$. The second of these is included in the term on the right-hand side of Eq. (10), $N(\alpha,\sim\gamma)$, provided that each member of the collection of specified objects, e.g., $N(\alpha,\sim\beta)$, does in fact have all three properties, e.g., γ or $\sim\gamma$, even though only two parameters are specified. Thus we are assuming a form of realism in the proof: that these objects have properties whether or not they are measured or specified.

Recalling Einstein's role in the origins of this controversy, and his long-standing objections to quantum theory, his self-stated "basic axiom" is interesting: "It is the postulation of a 'real world' which, so to speak, liberates the
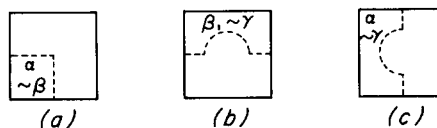
Table I. Example of a three double-valued parameter specification of a macroscopic physical object.

α = wood	β = black	γ = sphere
$\sim\alpha$ = metal	$\sim\beta$ = white	$\sim\gamma$ = cube

Fig. 3. Representing the number of objects with properties $(\alpha,\sim\beta)$, $(\beta,\sim\gamma)$, and $(\alpha,\sim\gamma)$.

Table II. Three-parameter specification for atomic spins for various polarizer orientations.

α = spin up, θ = 0°	β = spin up, θ = 45°	γ = spin up, θ = 90°
$\sim\alpha$ = spin down, θ = 0°	$\sim\beta$ = spin down, θ = 45°	$\sim\gamma$ = spin down, θ = 90°

'world' from the thinking and experiencing subject."[12] Thus the assumption of realism used in this proof seems consistent with Einstein's own feelings about the nature of the world.

Since Eq. (10) is true for any collection of classical objects, we may rewrite it in terms of the probability that in a random selection we find an object with specified properties:

$$\text{prob}(\alpha, \sim\beta) + \text{prob}(\beta, \sim\gamma) \geqslant \text{prob}(\alpha, \sim\gamma) \, . \qquad (11)$$

It is instructive to see how Eq. (11) is not expected to apply to quantum-mechanical spin measurements. Let the parameters α, β, and γ refer to measurements of atomic spins for polarizer orientations with respect to the z axis of 0°, 45°, and 90°, respectively, as in Table II. Then, if we construct stacks of Stern–Gerlach filters which only pass spin-up atoms, as in Fig. 4, we may rewrite Eq. (11) for the incident nonpolarized beam of atoms as

$$\text{prob}(\text{passes } \alpha, \text{ not } \beta) + \text{prob}(\text{passes } \beta, \text{ not } \gamma)$$
$$\geqslant \text{prob}(\text{passes } \alpha, \text{ not } \gamma) \, . \qquad (12)$$

This relation is not true because, for example, the measurement by polarizer α does effect the probability that the atom will subsequently be able to pass polarizer β by introducing uncontrollable phase factors into the wave function.[8] This is a form of the uncertainty principle: if the beam enters polarizer β first, one-half of the incident beam will pass, while if the beam enters α first, the probability of being able to subsequently pass β is changed. Put another way, the operators P'_α and P'_β do not commute, as can be easily seen by letting the final beam strike a polarizer oriented at 180° as in Fig. 5. In the first case shown, 3.1% of the incident beam will pass all three filters, while in the second case none will pass.

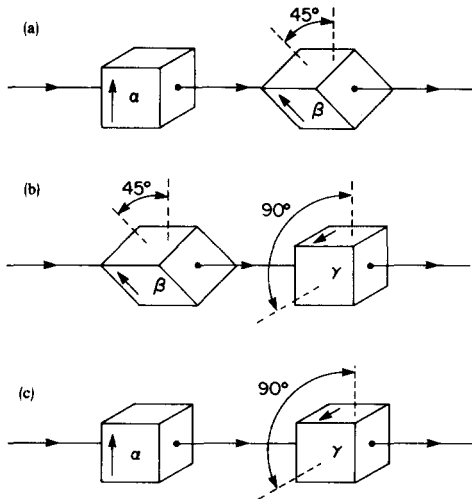However, the final step in this proof of the Bell inequa-

lity sidesteps this difficulty with the uncertainty principle. We again consider our collection of macroscopic classical objects for which Eq. (11) holds. We consider an experiment in which object 1 is picked randomly as before, and then a second object 2 is picked with each property opposite object 1. Then, for example, prob$(\alpha, \sim\beta)$ for object 1 is equal to the probability that object 1 has property α and the corresponding object 2 does have property β, prob$(1\alpha, 2\beta)$. Thus Eq. (11) becomes

$$\text{prob}(1\alpha, 2\beta) + \text{prob}(1\beta, 2\gamma) \geqslant \text{prob}(1\alpha, 2\gamma) \, . \qquad (13)$$

Now consider the application of Eq. (13) to correlated spin measurements as in Fig. 1(b). The fact that for the two objects each has all properties opposite the other corresponds to the fact that in the spin measurements if the two polarizers have the same orientation then the two atoms have opposite spins. Also, the two polarizers are measuring the spins at locations separated by a spacelike interval from the other polarizer's measurement. Thus the two polarizers cannot influence each other except by signals that travel faster than light. If such "superluminal" influences do not exist, therefore, we may expect Eq. (13) to apply to correlated spin measurements since the measurement by P'_1 cannot effect P'_2, and vice versa. Thus assigning parameters as before in Table II, we rewrite Eq. (13) for spin correlations:

$$\text{prob}(1 \text{ up } 0°, 2 \text{ up } 45°) + \text{prob}(1 \text{ up } 45°, 2 \text{ up } 90°)$$
$$\geqslant \text{prob} (1 \text{ up } 0°, 2 \text{ up } 90°) \, . \qquad (14)$$

This is the Bell inequality.

Each term in Eq. (14) can evaluated from quantum theory, Eq. (8). The result is

$$0.146 \geqslant 0.250 \qquad (15)$$

Thus the prediction of quantum mechanics violates the Bell inequality. In fact, this result is the same as attempting to apply Eq. (12) to the experiments of Fig. 4, a case where the uncertainty principle is expected to lead to a violation, as discussed.

We close this section by reviewing the assumptions we have made in deriving Eq. (14). We have assume separability: that no influence may propagate faster than the speed of light; this assumption is needed to "beat" the uncertainty principle. We have assumed realism: that objects have

Fig. 4. Three different arrays of Stern–Gerlach apparatuses. Each box passes only spin-up atoms.
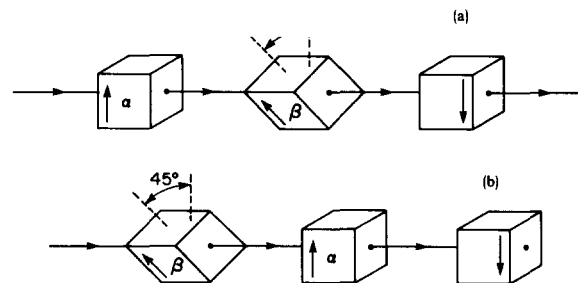

Fig. 5. Two different combinations of three Stern–Gerlach apparatuses. Each box passes only spin-up atoms.

properties such as spin components in various directions even if they are not measured. Finally, *inductive inference* has been used throughout the proof.

IV. STAPP'S PROOF

Stapp has derived the Bell inequality in a way that shows the conflict between quantum theory and common sense assumptions in a different way. Consider Stern–Gerlach apparatuses which measure the spin of the atoms, assigning a value of $+1$ for spin up, -1 for spin down. We then calculate the product $p'_1 p'_2$ of these values for the two atoms of the system in Fig. 1(b). Imagine we do this for N pairs of atoms, and denote the result of the jth measurement as $p'_{1j} p'_{2j}$. We consider four different angle combinations for the two polarizers:

$$
\begin{aligned}
\theta_1 &= 0° \quad \text{and} \quad \theta_2 = 0°, \\
\theta_1 &= 90° \quad \text{and} \quad \theta_2 = 0°, \\
\theta_1 &= 0° \quad \text{and} \quad \theta_2 = 135°, \\
\theta_1 &= 90° \quad \text{and} \quad \theta_2 = 135°.
\end{aligned}
\tag{16}
$$

For generality, label each eigenvalue of the polarization measurement with both angles, i.e., $p'_{1j}(\theta_1,\theta_2)$ and $p'_{2j}(\theta_1,\theta_2)$. Then for large N, the result of doing the four experiments of Eq. (16) can be calculated from quantum theory, Eq. (9):

$$
(1/N) \sum_{j=1}^{N} p'_{1j}(0°,0°) p'_{2j}(0°,0°) = -1,
$$

$$
(1/N) \sum_{j=1}^{N} p'_{1j}(90°,0°) p'_{2j}(90°,0°) = 0,
$$

$$
(1/N) \sum_{j=1}^{N} p'_{1j}(0°,135°) p'_{2j}(0°,135°) = 1/\sqrt{2},
$$

$$
(1/N) \sum_{j=1}^{N} p'_{1j}(90°,135°) p'_{2j}(90°,135°) = -1/\sqrt{2}.
\tag{17}
$$

So these are the predictions of quantum theory.

Now we make two assumptions which will lead to a conflict with Eq. (17). First we assume separability. This means that p'_{1j} should not depend on θ_2, and p'_{2j} should not depend on θ_1, since the two measurements are done at different spacetime locations separated by a spacelike interval. Thus the jth measurement by, say, P'_1 is independent of the orientation of P'_2.

The second assumption is that the result of the jth measurement by, say, P'_1 is not completely probabilistic and will be the same in all experimental runs with identical initial conditions. This second assumption in particular seems consistent with another of Einstein's "inner voices" about the nature of the world: God does not play dice with the universe.[13]

From these two assumptions, we write

$$
\begin{aligned}
p'_{1j}(0°,0°) &= p'_{1j}(0°,135°) \equiv p'_{1j}, \\
p'_{1j}(90°,0°) &= p'_{1j}(90°,135°) \equiv p''_{1j}, \\
p'_{2j}(0°,0°) &= p'_{2j}(90°,0°) \equiv p'_{2j}, \\
p'_{2j}(0°,135°) &= p'_{2j}(90°,135°) \equiv p''_{2j}.
\end{aligned}
\tag{18}
$$

As a matter of simple algebra, it is easy to show that Eq. (18) is not consistent with quantum theory, Eq. (17). We shall not duplicate the ten lines of algebra that are spelled out in Ref. 4; it suffices to say that by inserting Eq. (18) into

Eq. (17) one derives the result

$$
1 \geqslant \sqrt{2}.
\tag{19}
$$

V. EXPERIMENTAL TESTS AND DISCUSSION

In Secs. III and IV we had some unusual appearing angles (45°, 135°, etc.) for the orientations of the Stern–Gerlach magnets. The reason is that for angles of 0°, 90°, and 180° there is no conflict between the predictions of quantum mechanics and the Bell inequality. Further, it should be pointed out that an identical but more difficult analysis than above can be carried out for photons with total helicity zero,[14] giving similar discrepancies except for angles for the polarizers of 0°, 45°, and 90°.

The predictions of quantum mechanics have been experimentally tested for photons and protons at angles where discrepancies with the Bell inequality are expected,[15] and although these experiments are very difficult it is fair to say that the quantum correlations have been confirmed. This, of course, does not necessarily mean quantum mechanics is correct; it does, however, mean that at least some of the Bell assumptions of Secs. III and IV are wrong.

Both proofs above used an assumption of separability: that the two polarizers P'_1 and P'_2 act independently. If this assumption is not correct, then some sort of "superluminal" signal connecting the two polarizers is possible. Notice, however, that this does not necessarily imply a mechanism by which information may be transmitted from P'_1 to P'_2 at speeds greater than c. The effect that violates the Bell inequality only manifests in the correlations of the two detectors; each detector individually appears to find that one-half of the beam of atoms are spin up, one-half are spin down, and that which case is true for a particular atom is random.[16]

Another feature of this separability assumption is that the experimental tests have always involved setting the angles of the polarizers, and then later taking data. Thus it is at least conceivable that the polarizers have plenty of time to "know" each other's orientation. A stricter test would involve setting the angles of the polarizers after the atoms have left the source but before they reach the polarizers; Aspect is currently engaged in just such an experiment.[17]

The proof of d'Espagnat had two other assumptions: realism and inductive inference. A potential conflict between quantum theory and the assumption of an external reality independent of its observation has been suggested before.[18] Wheeler has summarized one view of the situation in quantum theory as, "No elementary phenomenon is a phenomenon until it is an observed phenomenon."[19] Thus d'Espagnat subtitled Ref. 3, "The doctrine that the world is made up of objects whose existence is independent of human consciousness turns out to be in conflict with quantum mechanics and with facts established by experiment." This, however, is only one of the possibilities, for there are two other assumptions in d'Espagnat's proof.[20]

As for d'Espagnat's assumption of the validity of inductive reasoning, the whole question of the validity of logic, and its appropriateness in quantum mechanics, is controversial.[21] To briefly illustrate one problem in the inductive method we retell a story which uses Mills' rules for induction completely correctly:

A favorite example used by critics of the Method of Agreement is the case of the Scientific Drinker, who was extremely fond of liquor and got drunk every night

of the week. He was ruining his health, and his few remaining friends pleaded with him to stop. Realizing himself that he could not go on, he resolved to conduct a careful experiment to discover the exact cause of his frequent inebriations. For five nights in a row he collected instances of a given phenomenon, the antecedent circumstances being respectively scotch and soda, bourbon and soda, brandy and soda, rum and soda, and gin and soda. Then using the Method of Agreement, he swore a solemn oath never to touch soda again![22]

Thus, at the least, rules of logical inference must be applied with caution. Note the "hidden variable" in the above example.

The proof by Stapp uses different assumptions than d'Espagnat. Though they both assume separability, Stapp's proof assumed a nonprobabilistic structure to the spin measurements. In particular we assumed that the result of the jth measurement by, say, P_1' would be the same in different experimental runs, which is not the same as saying that an atom has a definite spin before it is measured by P_1'. Thus the conflict between the Bell inequality and the experimental tests is summarized by Stapp: "No theory that predicts individual results that conform to the contingent predictions of quantum theory can be local."[23]

VI. CONCLUSION

Stapp has called the Bell theorem "the most profound discovery of science."[24] The consequences are in fact extremely deep.[25] We see that in either form of the proof presented above, some very general assumptions about the nature of the world are at least partly incorrect. It is amusing to speculate on the common ground of d'Espagnat's assumptions and Stapp's; it may not be entirely the separability assumption. At any rate it appears that these sorts of assumptions are close to many people's common sense. We thus close with Einstein's perceptive definition of common sense: prejudices acquired at an early age.

ACKNOWLEDGMENTS

My understanding of Bell's theorem has been greatly helped by N. Herbert, whose patience is gratefully acknowledged. Much of this material was first presented to the Canadian Association of Natural Philosophers,[26] various of my classes,[27] and the "Beer and Physics" group at the University of Toronto; their feedback has been extremely useful. Dr. G. Krebs read early forms of the manuscript, and his suggestions have been helpful to me.

APPENDIX

The wave function of the system is

$$\Psi = (1/\sqrt{2})[U_+(1)U_-(2) - U_-(1)U_+(2)] . \qquad (20)$$

Consider an alternative wave function:

$$\Psi' = (1/\sqrt{2})[U_+(1)U_-(2) + U_-(1)U_+(2)] . \qquad (21)$$

This wave function also describes a system where if atom 1 is spin up with respect to the z axis than atom 2 is spin down, and vice versa. But consider a measurement of the x component of the spins. We adopt the standard spinor notation:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

$$\qquad (22)$$

$$U_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ,$$

$$U_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

The eigenvectors of σ_x are

$$V_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} ,$$

$$\qquad (23)$$

$$V_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} ,$$

which in terms of the eigenvectors of σ_z are

$$V_+ = (1/\sqrt{2})(U_+ + U_-) ,$$

$$\qquad (24)$$

$$V_- = (1/\sqrt{2})(U_+ - U_-) .$$

Inserting Eq. (24) for both atom 1 and atom 2 into the wave function for the system, Eq. (20), gives

$$\Psi = (1/\sqrt{2})[V_+(1)V_-(2) - V_-(1)V_+(2)] , \qquad (25)$$

but

$$\Psi' = (1/\sqrt{2})[V_+(1)V_+(2) + V_-(1)V_-(2)] . \qquad (26)$$

Thus the result of measuring the x components of the spin for the system described by Ψ gives one atom spin up, the other spin down; the same measurement on the system described by Ψ' gives both atoms either spin up or spin down. Thus Ψ' represents a total spin of $\hbar$, but with zero value of the z component. Note, then, that the total spin of the system is an interference effect between the two terms in Ψ.

[1] J. S. Bell, Physics **1**, 195 (1964).

[2] A. Peres, Am. J. Phys. **46**, 745 (1978).

[3] B. d'Espagnat, Sci. Am. **241**, 158 (Nov. 1979).

[4] H. P. Stapp, Found. Phys. **9**, 1 (1979).

[5] R. P. Feynman, R. B. Leighton, and J. Sands, *The Feynman Lectures in Physics* (Addison-Wesley, Toronto, 1963), Vol. III.

[6] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[7] J. H. McGrath, Int. J. Theor. Phys. **17**, 557 (1978).

[8] D. Bohm, *Quantum Theory* (Prentice-Hall, Englewood Cliffs, NJ, 1951), pp. 611–623.

[9] See, for example, Ref. 5, Chap. 5-6.

[10] J. S. Bell, "Bertlmann's socks and the nature of reality," CERN Preprint TH.2926-CERN, July 1980 (unpublished).

[11] No objects with either properties $(\alpha, \sim\beta, \gamma)$ or $(\sim\alpha, \beta, \sim\gamma)$.

[12] G. Holton, *The Scientific Imagination* (Cambridge University, New York, 1978), p. 91.

[13] This objection of Einstein's to quantum mechanics goes back to 1926. See, e.g., M. Born, *The Born–Einstein Letters* (Walker, New York, 1971), p. 91.

[14] J. Clauser, M. Horne, A. Shimony, and R. Holt, Phys. Rev. Lett. **23**, 880 (1969).

[15] See, e.g., Ref. 3, or J. F. Clauser and A. Shimony, Rep. Prog. Phys. **41**, 1881 (1978).

[16] See, for example, B. d'Espagnat, Phys. Rev. D **11**, 1424 (1975). A discussion of the difference between cause, information, and influence is B. d'Espagnat, Found. Phys. **11**, 205 (1981).

[17] A. Aspect, Phys. Rev. D **14**, 1944 (1976). This experiment is also discussed in Refs. 3 and 10.

[18]See, for example, D. Harrison, Am. J. Phys. **47**, 576 (1979) and references cited there.

[19]*Problems in the Foundations of Physics*, edited by G. Toraldo di Francia (North-Holland, New York, 1980), Course LXXII, p. 395.

[20]The letter of V. F. Weisskopf and d'Espagnat's reply in Sci. Am. **242**, 8 (May 1980) discuss this subtitle further.

[21]The validity of logic itself may be explored in D. Hofstadter, *Gödel, Escher, Bach* (Basic, New York, 1979) or M. Kline, *Mathematics* (Oxford University, New York, 1980). A discussion of logic in quantum mechanics may be found in G. Zukav, *The Dancing Wu Li Masters*

(Bantam, New York, 1980), pp. 255ff.

[22]I. Copi, *Introduction to Logic*, 2nd ed. (Macmillan, New York, 1961), pp. 394–395.

[23]H. P. Stapp, Found. Phys. **10**, 767 (1980). Stapp prefers to state, "No process that selects observations that conform to the contingent predictions of quantum theory can be local."

[24]H. P. Stapp, Nuovo Cimento **29B**, 271 (1975).

[25]See, for example, D. J. Bohm and B. J. Hiley, Found. Phys. **5**, 93 (1975).

[26]D. Harrison and J. D. Prentice, Am. J. Phys. **48**, 799 (1980).

[27]D. Harrison, Am. J. Phys. **47**, 779 (1979).

Relativistic acoustic Doppler effect

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The formulation of the relativistic acoustic Doppler effect is considered in detail. Two approaches were developed, both leading to the same result. The first method parallels the derivation of the classical formula, using Lorentz velocity transformation and time dilation effects when required. The second method is an intuitive approach done in the frame of the medium, clearly showing that the relativistic correction terms arise solely from time dilation effects; the rest of the derivation is entirely classical. It is demonstrated that the relativistic deviation is exactly zero when the source and observer speeds are equal, even if the two motions are oppositely directed. Finally, a numerical estimate reveals that the correction terms amount to only a few parts in 10^{15} for ordinary acoustic velocities.

I. INTRODUCTION

Bright undergraduate physics students, after having studied the relativistic optical Doppler effect,[1] occasionally ask about the relativistically correct formulation of the *acoustic* Doppler effect. A pragmatic reply is that the correction is negligible because, in acoustics, the velocities involved are of the order of the speed of sound in air (about 340 m/s), which is six orders of magnitude less than the speed of light. Since relativisitic effects become apparent at speeds approaching that of light, it can be stated with assurance that the classical result is valid. However, the persistent purist, on strictly academic grounds, may demand a derivation of the correct acoustic formula. Becoming curious myself, I proceeded to study the problem in detail, developing two different approaches to the solution, both of which happily yield the same result. The deviation from the classical result, due to relativistic effects, can then be investigated, providing a quantitative answer to the student's query as well as an interesting application of special relativity.

II. DERIVATION OF THE CLASSICAL FORMULA

To begin, it is instructive to review the development of the classical formula, which will then be used as a foundation for the exact derivations.[2]

Consider the frame of reference in which the acoustic medium is at rest. A source of sound waves moves to the right with velocity v_s, while an observer moves with velocity $- v_0$ to the left, along the line joining source and observer. The wave speed u is the speed of sound in the frame of the medium. The wave frequency as seen by the observer is higher than the source frequency for two reasons: (i) the wavelength of the sound is shortened because the source moves during the time in which the wave is radiated, causing a "bunching" effect as successive waves are emitted; (ii) the moving observer intercepts more waves per unit time than does a stationary observer because the former is moving into the radiated waves, so that the apparent acoustic velocity is now $u + v_0$, resulting in proportionately more waves striking the observer's detector.

A derivation of the classical formula proceeds as follows. As mentioned above, the moving observer sees an apparent acoustic velocity of $u + v_0$. The apparent frequency, as see by the observer, is then

$$f' = (u + v_0)/\lambda ', \tag{1}$$

where $\lambda '$ is the wavelength seen by the observer. This is not equal to the source wavelength λ (measured for a stationary source) because of the "bunching" effect previously discussed. An expression for $\lambda '$ is deduced from the following argument. From the observer's frame, the apparent source velocity is $v_s + v_0$, while the time between successive wave emissions is $1/f_0$. In this time interval, the source travels a distance $(v_s + v_0)/f_0$, while the wave travels a distance $(u + v_0)/f_0$. The difference between these distances is the apparent wavelength $\lambda '$ seen in the observer's frame; i.e.,