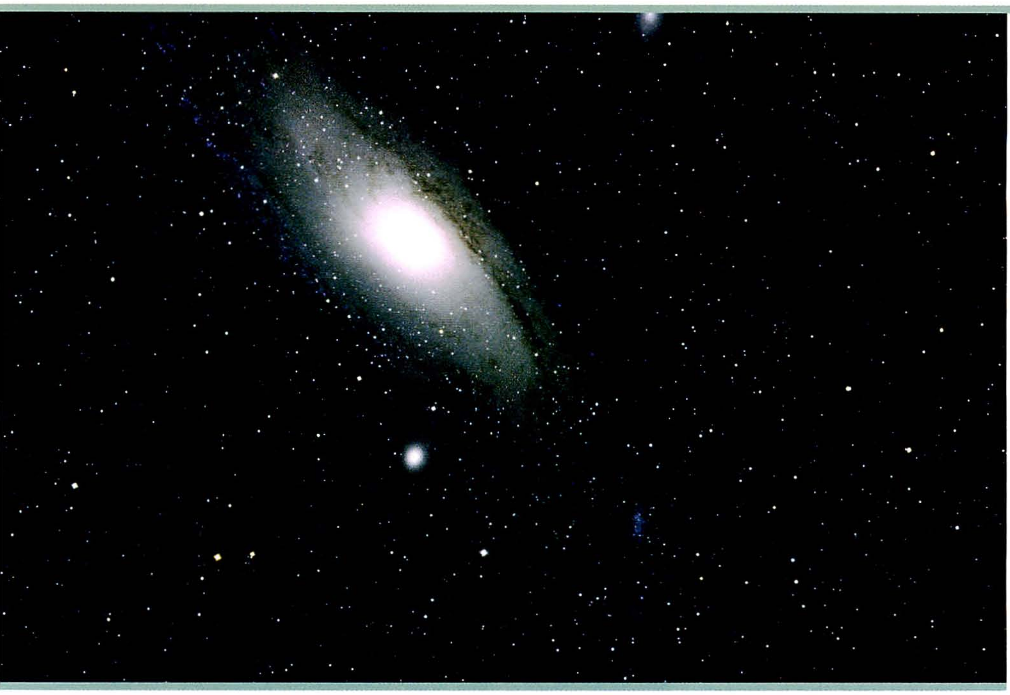
CHAPTER

39

Relativity



THE ANDROMEDA GALAXY BY MEASURING THE FREQUENCY OF THE LIGHT COMING TO US FROM DISTANT OBJECTS, WE ARE ABLE TO DETERMINE HOW FAST THESE OBJECTS ARE APPROACHING TOWARD US OR RECEDING FROM US.

? Have you wondered how the frequency of the light enables us to determine the speed of recession of a distant galaxy? This is discussed in Example 39-5.

The theory of relativity consists of two rather different theories, the special theory and the general theory. The special theory, developed by Albert Einstein and others in 1905, concerns the comparison of measurements made in different inertial reference frames moving with constant velocity relative to one another. Its consequences, which can be derived with a minimum of mathematics, are applicable in a wide variety of situations encountered in physics and in engineering. On the other hand, the general theory, also developed by Einstein and others around 1916, is concerned with accelerated reference frames and gravity. A thorough understanding of the general theory requires sophisticated mathematics, and the applications of this theory are chiefly in the area of gravitation. The general theory is of great importance in cosmology, but it is rarely encountered in other areas of physics or in engineering. The general theory is used, however, in the engineering of the Global Positioning System (GPS).†

† The satellites used in GPS contain atomic clocks.

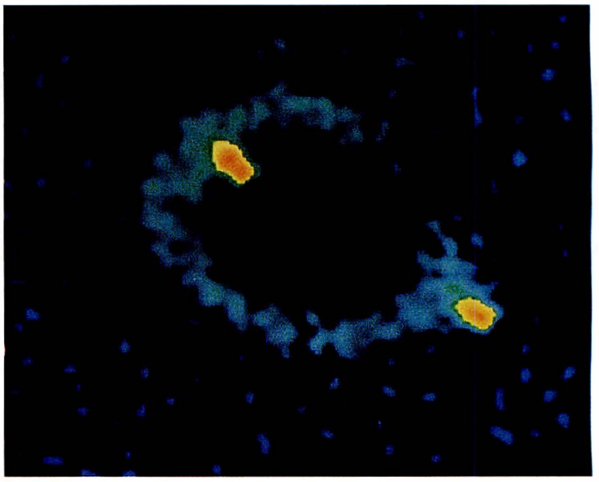
➢In this chapter, we concentrate on the special theory (often referred to as *special relativity*). General relativity will be discussed briefly near the end of the chapter.

39-1 Newtonian Relativity

Newton's first law does not distinguish between a particle at rest and a particle moving with constant velocity. If there is no net external force acting, the particle will remain in its initial state, either at rest or moving with its initial velocity. A particle at rest relative to you is moving with constant velocity relative to an observer who is moving with constant velocity relative to you. How might we distinguish whether you and the particle are at rest and the second observer is moving with constant velocity, or the second observer is at rest and you and the particle are moving?

Let us consider some simple experiments. Suppose we have a railway boxcar moving along a straight, flat track with a constant velocity v. We note that a ball at rest in the boxcar remains at rest. If we drop the ball, it falls straight down, relative to the boxcar, with an acceleration g due to gravity. Of course, when viewed from the track the ball moves along a parabolic path because it has an initial velocity v to the right. No mechanics experiment that we can do—measuring the period of a pendulum, observing the collisions between two objects, or whatever—will tell us whether the boxcar is moving and the track is at rest or the track is moving and the boxcar is at rest. If we have a coordinate system attached to the track and another attached to the boxcar, Newton's laws hold in either system.

A set of coordinate systems at rest relative to each other is called a *reference frame*. A reference frame in which Newton's laws hold is called an *inertial reference frame*.† All reference frames moving at constant velocity relative to an inertial reference frame are also inertial reference frames. If we have two inertial reference frames moving with constant velocity relative to each other, there are no mechanics experiments that can tell us which is at rest and which is moving or if they are both moving. This result is known as the principle of **Newtonian relativity:**



This ring-like structure of the radio source MG1131 + 0456 is thought to be due to *gravitational lensing*, first proposed by Albert Einstein in 1936, in which a source is imaged into a ring by a large, massive object in the foreground.

Absolute motion cannot be detected.

PRINCIPLE OF NEWTONIAN RELATIVITY

This principle was well known by Galileo, Newton, and others in the seventeenth century. By the late nineteenth century, however, this view had changed. It was then generally thought that Newtonian relativity was not valid and that absolute motion could be detected in principle by a measurement of the speed of light.

Ether and the Speed of Light

We saw in Chapter 15 that the velocity of a wave depends on the properties of the medium in which the wave travels and not on the velocity of the source of the waves. For example, the velocity of sound relative to still air depends on the temperature of the air. Light and other electromagnetic waves (radio, X rays, etc.) travel through a vacuum with a speed $c \approx 3 \times 10^8$ m/s that is predicted by James Clerk Maxwell's equations for electricity and magnetism. But what is this speed

† Reference frames were first discussed in Section 2-1. Inertial reference frames were also discussed in Section 4-1.

relative to? What is the equivalent of still air for a vacuum? A proposed medium for the propagation of light was called the *ether;* it was thought to pervade all space. The velocity of light relative to the ether was assumed to be c, as predicted by Maxwell's equations. The velocity of any object relative to the ether was considered its absolute velocity.

Albert Michelson, first in 1881 and then again with Edward Morley in 1887, set out to measure the velocity of the earth relative to the ether by an ingenious experiment in which the velocity of light relative to the earth was compared for two light beams, one in the direction of the earth's motion relative to the sun and the other perpendicular to the direction of the earth's motion. Despite painstakingly careful measurements, they could detect no difference. The experiment has since been repeated under various conditions by a number of people, and no difference has ever been found. The absolute motion of the earth relative to the ether cannot be detected.

39-2 Einstein's Postulates

In 1905, at the age of 26, Albert Einstein published a paper on the electrodynamics of moving bodies.† In this paper, he postulated that absolute motion cannot be detected by any experiment. That is, there is no ether. The earth can be considered to be at rest and the velocity of light will be the same in any direction.‡ His theory of special relativity can be derived from two postulates. Simply stated, these postulates are as follows:

Postulate 1: Absolute uniform motion cannot be detected.

Postulate 2: The speed of light is independent of the motion of the source.

EINSTEIN'S POSTULATES

Postulate 1 is merely an extension of the Newtonian principle of relativity to include all types of physical measurements (not just those that are mechanical). Postulate 2 describes a common property of all waves. For example, the speed of sound waves does not depend on the motion of the sound source. The sound waves from a car horn travel through the air with the same velocity independent of whether the car is moving or not. The speed of the waves depends only on the properties of the air, such as its temperature.

Although each postulate seems quite reasonable, many of the implications of the two postulates together are quite surprising and contradict what is often called common sense. For example, one important implication of these postulates is that every observer measures the same value for the speed of light independent of the relative motion of the source and the observer. Consider a light source S and two observers, R_1 at rest relative to S and R_2 moving toward S with speed v, as shown in Figure 39-1a. The speed of light measured by R_1 is $c = 3 \times 10^8$ m/s. What is the speed measured by R_2? The answer is *not* $c + v$. By postulate 1, Figure 39-1a is equivalent to Figure 39-1b, in which R_2 is at rest and the source S and R_1 are moving with speed v. That is, since absolute motion cannot be detected, it is not possible to say which is really moving and which is at rest. By postulate 2, the speed of light from a moving source is independent of the

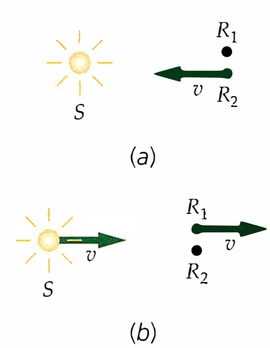


FIGURE 39-1 (*a*) A stationary light source S and a stationary observer R_1, with a second observer R_2 moving toward the source with speed v. (*b*) In the reference frame in which the observer R_2 is at rest, the light source S and observer R_1 move to the right with speed v. If absolute motion cannot be detected, the two views are equivalent. Since the speed of light does not depend on the motion of the source, observer R_2 measures the same value for that speed as observer R_1.

† *Annalen der Physik,* vol. 17, 1905, p. 841. For a translation from the original German, see W. Perrett and G. B. Jeffery (trans.), *The Principle of Relativity: A Collection of Original Memoirs on the Special and General Theory of Relativity* by H. A. Lorentz, A. Einstein, H. Minkowski, and W. Weyl, Dover, New York, 1923.

‡ Einstein did not set out to explain the results of the Michelson–Morley experiment. His theory arose from his considerations of the theory of electricity and magnetism and the unusual property of electromagnetic waves that they propagate in a vacuum. In his first paper, which contains the complete theory of special relativity, he made only a passing reference to the Michelson–Morley experiment, and in later years he could not recall whether he was aware of the details of this experiment before he published his theory.

motion of the source. Thus, looking at Figure 39-1*b*, we see that R_2 measures the speed of light to be c, just as R_1 does. This result is often considered as an alternative to Einstein's second postulate:

Postulate 2 (alternate): Every observer measures the same value c for the speed of light.

This result contradicts our intuitive ideas about relative velocities. If a car moves at 50 km/h away from an observer and another car moves at 80 km/h in the same direction, the velocity of the second car relative to the first car is 30 km/h. This result is easily measured and conforms to our intuition. However, according to Einstein's postulates, if a light beam is moving in the direction of the cars, observers in both cars will measure the same speed for the light beam. Our intuitive ideas about the combination of velocities are approximations that hold only when the speeds are very small compared with the speed of light. Even in an airplane moving with the speed of sound, to measure the speed of light accurately enough to distinguish the difference between the results c and $c + v$, where v is the speed of the plane, would require a measurement with six-digit accuracy.

39-3 The Lorentz Transformation

Einstein's postulates have important consequences for measuring time intervals and space intervals, as well as relative velocities. Throughout this chapter, we will be comparing measurements of the positions and times of events (such as lightning flashes) made by observers who are moving relative to each other. We will use a rectangular coordinate system xyz with origin O, called the S reference frame, and another system $x'y'z'$ with origin O', called the S' frame, that is moving with a constant velocity $\vec{v}$ relative to the S frame. Relative to the S' frame, the S frame is moving with a constant velocity $-\vec{v}$. For simplicity, we will consider the S' frame to be moving along the x axis in the positive x direction relative to S. In each frame, we will assume that there are as many observers as are needed who are equipped with measuring devices, such as clocks and metersticks, that are identical when compared at rest (see Figure 39-2).

We will use Einstein's postulates to find the general relation between the coordinates x, y, and z and the time t of an event as seen in reference frame S and the coordinates x', y', and z' and the time t' of the same event as seen in reference frame S', which is moving with uniform velocity relative to S. We assume that the origins are coincident at time $t = t' = 0$. The classical relation, called the **Galilean transformation,** is

$$x = x' + vt', \qquad y = y', \qquad z = z', \qquad t = t' \tag{39-1a}$$

GALILEAN TRANSFORMATION

The inverse transformation is

$$x' = x - vt, \qquad y' = y, \qquad z' = z, \qquad t' = t \tag{39-1b}$$

These equations are consistent with experimental observations as long as v is much less than c. They lead to the familiar classical addition law for velocities. If a particle has velocity $u_x = dx/dt$ in frame S, its velocity in frame S' is

$$u'_x = \frac{dx'}{dt'} = \frac{dx'}{dt} = \frac{dx}{dt} - v = u_x - v \tag{39-2}$$

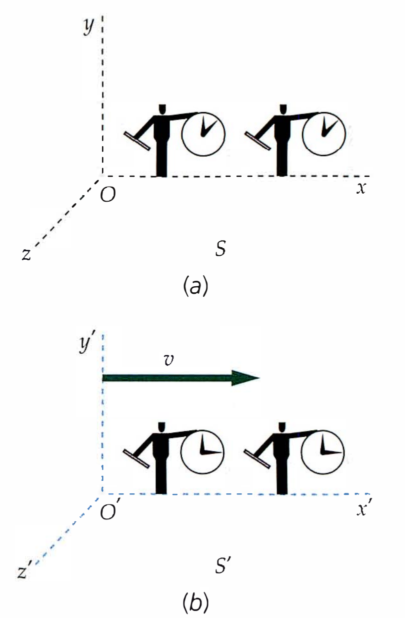


FIGURE 39-2 Coordinate reference frames S and S' moving with relative speed v. In each frame, there are observers with metersticks and clocks that are identical when compared at rest.

If we differentiate this equation again, we find that the acceleration of the particle is the same in both frames:

$$a_x = \frac{du_x}{dt} = \frac{du'_x}{dt'} = a'_x$$

It should be clear that the Galilean transformation is not consistent with Einstein's postulates of special relativity. If light moves along the x axis with speed $u'_x = c$ in S', these equations imply that the speed in S' is $u_x = c + v$ rather than $u_x = c$, which is consistent with Einstein's postulates and with experiment. The classical transformation equations must therefore be modified to make them consistent with Einstein's postulates. We will give a brief outline of one method of obtaining the relativistic transformation.

We assume that the relativistic transformation equation for x is the same as the classical equation (Equation 39-1*a*) except for a constant multiplier on the right side. That is, we assume the equation is of the form

$$x = \gamma(x' + vt') \tag{39-3}$$

where γ is a constant that can depend on v and c but not on the coordinates. The inverse transformation must look the same except for the sign of the velocity:

$$x' = \gamma(x - vt) \tag{39-4}$$

Let us consider a light pulse that starts at the origin of S at $t = 0$. Since we have assumed that the origins are coincident at $t = t' = 0$, the pulse also starts at the origin of S' at $t' = 0$. Einstein's postulates require that the equation for the x component of the wave front of the light pulse is $x = ct$ in frame S and $x' = ct'$ in frame S'. Substituting ct for x and ct' for x' in Equation 39-3 and Equation 39-4, we obtain

$$ct = \gamma(ct' + vt') = \gamma(c + v)t' \tag{39-5}$$

and

$$ct' = \gamma(ct - vt) = \gamma(c - v)t \tag{39-6}$$

We can eliminate the ratio t'/t from these two equations and determine γ. Thus,

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{39-7}$$

Note that γ is always greater than 1, and that when v is much less than c, $\gamma \approx 1$. The relativistic transformation for x and x' is therefore given by Equation 39-3 and Equation 39-4, with γ given by Equation 39-7. We can obtain equations for t and t' by combining Equation 39-3 with the inverse transformation given by Equation 39-4. Substituting $x = \gamma(x' + vt')$ for x in Equation 39-4, we obtain

$$x' = \gamma[\gamma(x' + vt') - vt] \tag{39-8}$$

which can be solved for t in terms of x' and t'. The complete relativistic transformation is

$$x = \gamma(x' + vt'), \qquad y = y', \qquad z = z' \tag{39-9}$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \tag{39-10}$$

LORENTZ TRANSFORMATION

The inverse transformation is

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z \tag{39-11}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{39-12}$$

The transformation described by Equation 39-9 through Equation 39-12 is called the **Lorentz transformation.** It relates the space and time coordinates $x, y, z,$ and t of an event in frame S to the coordinates x', y', z', and t' of the same event as seen in frame S', which is moving along the x axis with speed v relative to frame S.

We will now look at some applications of the Lorentz transformation.

Time Dilation

Consider two events that occur at a single point x_0' at times t_1' and t_2' in frame S'. We can find the times t_1 and t_2 for these events in S from Equation 39-10. We have

$$t_1 = \gamma\left(t_1' + \frac{vx_0'}{c^2}\right)$$

and

$$t_2 = \gamma\left(t_2' + \frac{vx_0'}{c^2}\right)$$

so

$$t_2 - t_1 = \gamma(t_2' - t_1')$$

The time between events that happen at the *same place* in a reference frame is called **proper time** t_p. In this case, the time interval $t_2' - t_1'$ measured in frame S' is proper time. The time interval Δt measured in any other reference frame is always longer than the proper time. This expansion is called **time dilation:**

$$\Delta t = \gamma\, \Delta t_p \tag{39-13}$$

TIME DILATION

SPATIAL SEPARATION AND TEMPORAL SEPARATION OF TWO EVENTS — EXAMPLE 39-1

Two events occur at the same point x_0' at times t_1' and t_2' in frame S', which is traveling at speed v relative to frame S. (*a*) What is the spatial separation of these events in frame S? (*b*) What is the temporal separation of these events in frame S?

PICTURE THE PROBLEM The spatial separation in S is $x_2 - x_1$, where x_2 and x_1 are the coordinates of the events in S, which are found using Equation 39-9.

(*a*) 1. The position x_1 in S is given by Equation 39-9 with $x_1' = x_0'$: $\quad x_1 = \gamma(x_0' + vt_1')$

2. Similarly, the position x_2 in S is given by: $\quad x_2 = \gamma(x_0' + vt_2')$

3. Subtract to find the spatial separation: $\quad \Delta x = x_2 - x_1 = \gamma v(t_2' - t_1') = \boxed{\dfrac{v(t_2' - t_1')}{\sqrt{1 - (v^2/c^2)}}}$

(*b*) Using the time dilation formula, relate the two time intervals. The two events occur at the same place in S', so the proper time between the two events is $\Delta t_p = t_2' - t_1'$: $\quad \Delta t = t_2 - t_1 = \gamma(t_2' - t_1') = \boxed{\dfrac{(t_2' - t_1')}{\sqrt{1 - (v^2/c^2)}}}$

REMARKS Dividing the Part (*a*) result by the Part (*b*) result gives $\Delta x/\Delta t = v$. The spatial separation of these two events in S is the distance a fixed point, such as x_0' in S', moves in S during the time interval between the events in S.

We can understand time dilation directly from Einstein's postulates without using the Lorentz transformation. Figure 39-3*a* shows an observer A' a distance D from a mirror. The observer and the mirror are in a spaceship that is at rest in frame S'. The observer explodes a flash gun and measures the time interval $\Delta t'$ between the original flash and his seeing the return flash from the mirror. Because light travels with speed c, this time is

$$\Delta t' = \frac{2D}{c}$$

We now consider these same two events, the original flash of light and the receiving of the return flash, as observed in reference frame S, in which observer A' and the mirror are moving to the right with speed v, as shown in Figure 39-3*b*.

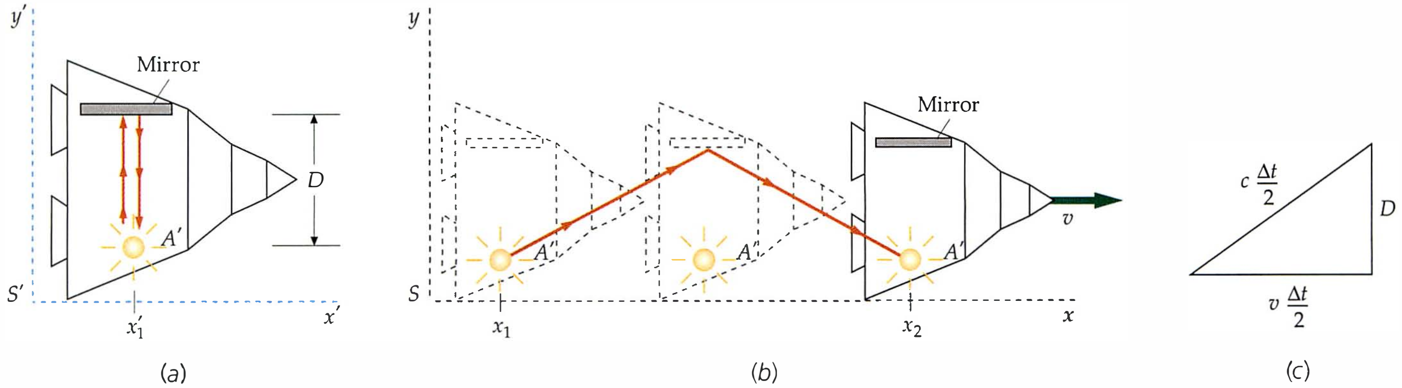


FIGURE 39-3 (*a*) Observer A' and the mirror are in a spaceship at rest in frame S'. The time it takes for the light pulse to reach the mirror and return is measured by A' to be $2D/c$. (*b*) In frame S, the spaceship is moving to the right with speed v. If the speed of light is the same in both frames, the time it takes for the light to reach the mirror and return is longer than $2D/c$ in S because the distance traveled is greater than $2D$. (*c*) A right triangle for computing the time Δt in frame S.

The events happen at two different places x_1 and x_2 in frame S. During the time interval Δt (as measured in S) between the original flash and the return flash, observer A' and his spaceship have moved a horizontal distance $v\,\Delta t$. In Figure 39-3*b*, we can see that the path traveled by the light is longer in S than in S'. However, by Einstein's postulates, light travels with the same speed c in frame S as it does in frame S'. Because light travels farther in S at the same speed, it takes longer in S to reach the mirror and return. The time interval in S is thus longer than it is in S'. From the triangle in Figure 39-3*c*, we have

$$\left(\frac{c\,\Delta t}{2}\right)^2 = D^2 + \left(\frac{v\,\Delta t}{2}\right)^2$$

or

$$\Delta t = \frac{2D}{\sqrt{c^2 - v^2}} = \frac{2D}{c}\frac{1}{\sqrt{1 - (v^2/c^2)}}$$

Using $\Delta t' = 2D/c$, we obtain

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - (v^2/c^2)}} = \gamma \Delta t'$$

HOW LONG IS A ONE-HOUR NAP? **EXAMPLE 39-2** **Try It Yourself**

Astronauts in a spaceship traveling at $v = 0.6c$ relative to the earth sign off from space control, saying that they are going to nap for 1 h and then call back. How long does their nap last as measured on the earth?

PICTURE THE PROBLEM Because the astronauts go to sleep and wake up at the same place in their reference frame, the time interval for their nap of 1 h as measured by them is proper time. In the earth's reference frame, they move a considerable distance between these two events. The time interval measured in the earth's frame (using two clocks located at those events) is longer by the factor γ.

Cover the column to the right and try these on your own before looking at the answers.

Steps	Answers
1. Relate the time interval measured on the earth Δt to the proper time Δt_p.	$\Delta t = \gamma \Delta t_p$
2. Calculate γ for $v = 0.6c$.	$\gamma = 1.25$
3. Substitute to calculate the time of the nap in the earth's frame.	$\Delta t = \gamma \Delta t_p = \boxed{1.25\text{ h}}$

EXERCISE If the spaceship is moving at $v = 0.8c$, how long would a 1 h nap last as measured on the earth? (*Answer* 1.67 h)

Length Contraction

A phenomenon closely related to time dilation is **length contraction.** The length of an object measured in the reference frame in which the object is at rest is called its **proper length** L_p. In a reference frame in which the object is moving, the measured length is shorter than its proper length. Consider a rod at rest in frame S' with one end at x_2' and the other end at x_1'. The length of the rod in this frame is its proper length $L_p = x_2' - x_1'$. Some care must be taken to find the length of the rod in frame S. In this frame, the rod is moving to the right with speed v, the speed of frame S'. The length of the rod in frame S is defined as $L = x_2 - x_1$, where x_2 is the position of one end at some time t_2, and x_1 is the position of the other end *at the same time* $t_1 = t_2$ as measured in frame S. Equation 39-11 is convenient to use to calculate $x_2 - x_1$ at some time t because it relates x and x' to t, whereas Equation 39-9 is not convenient because it relates x and x' to t':

$$x_2' = \gamma(x_2 - vt_2)$$

and

$$x_1' = \gamma(x_1 - vt_1)$$

Since $t_2 = t_1$, we obtain

$$x_2' - x_1' = \gamma(x_2 - x_1)$$

$$x_2 - x_1 = \frac{1}{\gamma}(x_2' - x_1') = (x_2' - x_1')\sqrt{1 - \frac{v^2}{c^2}}$$

or

$$L = \frac{1}{\gamma}L_p = L_p\sqrt{1 - \frac{v^2}{c^2}} \qquad 39\text{-}14$$

LENGTH CONTRACTION

Thus, the length of a rod is smaller when it is measured in a frame in which it is moving. Before Einstein's paper was published, Hendrik A. Lorentz and George F. FitzGerald tried to explain the null result of the Michelson–Morley experiment by assuming that distances in the direction of motion contracted by the amount given in Equation 39-14. This length contraction is now known as the **Lorentz–FitzGerald contraction.**

THE LENGTH OF A MOVING METERSTICK **EXAMPLE 39-3**

A stick that has a proper length of 1 m moves in a direction along its length with speed v relative to you. The length of the stick as measured by you is 0.914 m. What is the speed v?

PICTURE THE PROBLEM Since both L and L_p are given, we can find v directly from Equation 39-14.

1. Equation 39-14 relates the lengths L and L_p and the speed v:

$$L = L_p\sqrt{1 - \frac{v^2}{c^2}}$$

2. Solve for v:

$$v = c\sqrt{1 - \frac{L^2}{L_p^2}} = c\sqrt{1 - \frac{(0.914\ \text{m})^2}{(1\ \text{m})^2}} = \boxed{0.406c}$$

An interesting example of time dilation or length contraction is afforded by the appearance of muons as secondary radiation from cosmic rays. Muons decay according to the statistical law of radioactivity:

$$N(t) = N_0 e^{-t/\tau} \qquad 39\text{-}15$$

where N_0 is the original number of muons at time $t = 0$, $N(t)$ is the number remaining at time t, and τ is the mean lifetime, which is approximately 2 μs for muons at rest. Since muons are created (from the decay of pions) high in the atmosphere, usually several thousand meters above sea level, few muons should reach sea level. A typical muon moving with speed $0.9978c$ would travel only about 600 m in 2 μs. However, the lifetime of the muon measured in the earth's reference frame is increased by the factor $1/\sqrt{1 - (v^2/c^2)}$, which is 15 for this particular speed. The mean lifetime measured in the earth's reference frame is therefore 30 μs, and a muon with speed $0.9978c$ travels approximately 9000 m in this time. From the muon's point of view, it lives only 2 μs, but the atmosphere is rushing past it with a speed of $0.9978c$. The distance of 9000 m in the earth's frame is thus contracted to only 600 m in the muon's frame, as indicated in Figure 39-4.

It is easy to distinguish experimentally between the classical and relativistic predictions of the observation of muons at sea level. Suppose that we observe 10^8 muons at an altitude of 9000 m in some time interval with a muon detector. How many would we expect to observe at sea level in the same time interval? According to the nonrelativistic prediction, the time it takes for these muons to travel 9000 m is $(9000\text{ m})/(0.998c) \approx 30\ \mu\text{s}$, which is 15 lifetimes. Substituting $N_0 = 10^8$ and $t = 15\tau$ into Equation 39-15, we obtain

$$N = 10^8 e^{-15} = 30.6$$

We would thus expect all but about 31 of the original 100 million muons to decay before reaching sea level.

According to the relativistic prediction, the earth must travel only the contracted distance of 600 m in the rest frame of the muon. This takes only $2\ \mu\text{s} = 1\tau$. Therefore, the number of muons expected at sea level is

$$N = 10^8 e^{-1} = 3.68 \times 10^7$$

Thus, relativity predicts that we would observe 36.8 million muons in the same time interval. Experiments of this type have confirmed the relativistic predictions.

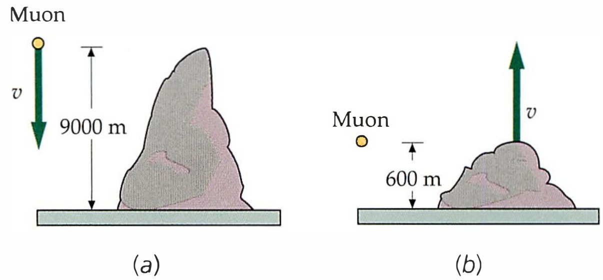


FIGURE 39-4 Although muons are created high above the earth and their mean lifetime is only about 2 μs when at rest, many appear at the earth's surface. (*a*) In the earth's reference frame, a typical muon moving at $0.998c$ has a mean lifetime of 30 μs and travels 9000 m in this time. (*b*) In the reference frame of the muon, the distance traveled by the earth is only 600 m in the muon's lifetime of 2 μs.

The Relativistic Doppler Effect

For light or other electromagnetic waves in a vacuum, a distinction between motion of source and receiver cannot be made. Therefore, the expressions we derived in Chapter 15 for the Doppler effect cannot be correct for light. The reason is that in that derivation, we assumed the time intervals in the reference frames of the source and receiver to be the same.

Consider a source moving toward a receiver with velocity v, relative to the receiver. If the source emits N electromagnetic waves in a time Δt_R (measured in the frame of the receiver), the first wave will travel a distance $c\,\Delta t_R$ and the source will travel a distance $v\,\Delta t_R$ measured in the frame of the receiver. The wavelength will be

$$\lambda' = \frac{c\,\Delta t_R - v\,\Delta t_R}{N}$$

The frequency f' observed by the receiver will therefore be

$$f' = \frac{c}{\lambda'} = \frac{c}{c - v}\frac{N}{\Delta t_R} = \frac{1}{1 - (v/c)}\frac{N}{\Delta t_R}$$

If the frequency of the source is f_0, it will emit $N = f_0\,\Delta t_S$ waves in the time Δt_S measured by the source. Then

$$f' = \frac{1}{1 - (v/c)}\frac{N}{\Delta t_R} = \frac{1}{1 - (v/c)}\frac{f_0\,\Delta t_S}{\Delta t_R} = \frac{f_0}{1 - (v/c)}\frac{\Delta t_S}{\Delta t_R}$$

Here Δt_S is the proper time interval (the first wave and the Nth wave are emitted at the same place in the source's reference frame). Times Δt_S and Δt_R are related by Equation 39-13 for time dilation:

$$\Delta t_R = \gamma\,\Delta t_S = \frac{\Delta t_S}{\sqrt{1 - (v^2/c^2)}}$$

Thus, when the source and the receiver are moving toward one another we obtain

$$f' = \frac{f_0}{1-(v/c)}\frac{1}{\gamma} = \frac{\sqrt{1-(v/c)^2}}{1-(v/c)}f_0 = \sqrt{\frac{1+(v/c)}{1-(v/c)}}f_0, \quad \text{approaching} \qquad \text{39-16}a$$

This differs from our classical equation only in the time-dilation factor. It is left as a problem (Problem 27) for you to show that the same results are obtained if the calculations are done in the reference frame of the source.

When the source and the receiver are moving away from one another, the same analysis shows that the observed frequency is given by

$$f' = \frac{\sqrt{1-(v/c)^2}}{1+(v/c)}f_0 = \sqrt{\frac{1-(v/c)}{1+(v/c)}}f_0, \quad \text{receding} \qquad \text{39-16}b$$

An application of the relativistic Doppler effect is the **redshift** observed in the light from distant galaxies. Because the galaxies are moving away from us, the light they emit is shifted toward the longer red wavelengths. The speed of the galaxies relative to us can be determined by measuring this shift.

CONVINCING THE JUDGE **EXAMPLE 39-4** **Put It in Context**

As part of a community volunteering option on your campus, you are spending the day shadowing two police officers. You have just had the excitement of pulling over a car that went through a red light. The driver claims that the red light looked green because the car was moving toward the stoplight, which shifted the wavelength of the observed light. You quickly do some calculations to see if the driver has a reasonable case or not.

PICTURE THE PROBLEM We can use the Doppler shift formula for approaching objects in Equation 39-16*a*. This will tell us the velocity, but we need to know the frequencies of the light. We can make good guesses for the wavelengths of red light and green light and use the definition of the speed of a wave $c = f\lambda$ to determine the frequencies.

1. The observer is approaching the light source, so we use the Doppler formula (Equation 39-16*a*) for approaching sources:

$$f' = \sqrt{\frac{1+(v/c)}{1-(v/c)}}f_0$$

2. Substitute c/λ for f, then simplify:

$$\frac{c}{\lambda'} = \sqrt{\frac{1+(v/c)}{1-(v/c)}}\frac{c}{\lambda_0}$$

$$\left(\frac{\lambda_0}{\lambda'}\right)^2 = \frac{1+(v/c)}{1-(v/c)}$$

3. Cross multiply and solve for v/c:

$$(\lambda_0)^2\left(1-\frac{v}{c}\right) = (\lambda')^2\left(1+\frac{v}{c}\right)$$

$$(\lambda_0)^2 - (\lambda')^2 = \left[(\lambda_0)^2 + (\lambda')^2\right]\left(\frac{v}{c}\right)$$

$$\frac{v}{c} = \frac{(\lambda_0)^2-(\lambda')^2}{(\lambda_0)^2+(\lambda')^2} = \frac{1-(\lambda'/\lambda_0)^2}{1+(\lambda'/\lambda_0)^2}$$

4. The values for the wavelengths for the colors of the visible spectrum can be found in Table 30-1. The wavelengths for red are 725 nm or longer, and the wavelengths for green are 675 nm or shorter. Solve for the speed needed to shift the wavelength from 725 nm to 675 nm:

$$\frac{\lambda'}{\lambda_0} = \frac{675\,\text{nm}}{725\,\text{nm}} = 0.931$$

$$\frac{v}{c} = \frac{1 - 0.931^2}{1 + 0.931^2} = 0.0713$$

$$v = 0.0713c = 2.14 \times 10^7\ \text{m/s} = 4.79 \times 10^7\ \text{mi/h}$$

5. This speed is beyond any possible speed for a car:

$\boxed{\text{The driver does not have a plausible case.}}$

FINDING SPEED FROM THE DOPPLER SHIFT

EXAMPLE 39-5 Try It Yourself

The longest wavelength of light emitted by hydrogen in the Balmer series is $\lambda_0 = 656$ nm. In light from a distant galaxy, this wavelength is measured to be $\lambda' = 1458$ nm. Find the speed at which the distant galaxy is receding from the earth.

Cover the column to the right and try these on your own before looking at the answers.

Steps / **Answers**

1. Use Equation 39-16*b* to relate the speed v to the received frequency f' and the emitted frequency f_0.

$$f' = \sqrt{\frac{1-(v/c)}{1+(v/c)}}\,f_0$$

2. Substitute $f' = c/\lambda'$ and $f_0 = c/\lambda_0$ and solve for v/c.

$$\frac{v}{c} = \frac{1-(\lambda_0/\lambda')^2}{1+(\lambda_0/\lambda')^2} = 0.664$$

$$v = \boxed{0.664c}$$

39-4 Clock Synchronization and Simultaneity

We saw in Section 39-3 that proper time is the time interval between two events that occur at the same point in some reference frame. It can therefore be measured on a single clock. (Remember, in each frame there is a clock at each point in space, and the time of an event in a given frame is measured by the clock at that point.) However, in another reference frame moving relative to the first, the same two events occur at different places, so two clocks are needed to record the times. The time of each event is measured on a different clock, and the interval is found by subtraction. This procedure requires that the clocks be **synchronized.** We will show in this section that

> Two clocks that are synchronized in one reference frame are typically not synchronized in any other frame moving relative to the first frame.

SYNCHRONIZED CLOCKS

Here is a corollary to this result:

> Two events that are simultaneous in one reference frame typically are not simultaneous in another frame that is moving relative to the first.†

SIMULTANEOUS EVENTS

† This is true *unless* the x coordinates of the two events are equal, where the x axis is parallel with the relative velocity of the two frames.

Comprehension of these facts usually resolves all relativity paradoxes. Unfortunately, the intuitive (and incorrect) belief that simultaneity is an absolute relation is difficult to overcome.

Suppose we have two clocks at rest at point A and point B a distance L apart in frame S. How can we synchronize these two clocks? If an observer at A looks at the clock at B and sets her clock to read the same time, the clocks will not be synchronized because of the time L/c it takes light to travel from one clock to another. To synchronize the clocks, the observer at A must set her clock ahead by the time L/c. Then she will see that the clock at B reads a time that is L/c behind the time on her clock, but she will calculate that the clocks are synchronized when she allows for the time L/c for the light to reach her. Any other observers in S (except those equidistant from the clocks) will see the clocks reading different times, but they will also calculate that the clocks are synchronized when they correct for the time it takes the light to reach them. An equivalent method for synchronizing two clocks would be for an observer C at a point midway between the clocks to send a light signal and for the observers at A and B to set their clocks to some prearranged time when they receive the signal.

We now examine the question of **simultaneity.** Suppose A and B agree to explode flashguns at t_0 (having previously synchronized their clocks). Observer C will see the light from the two flashes at the same time, and because he is equidistant from A and B, he will conclude that the flashes were simultaneous. Other observers in frame S will see the light from A or B first, depending on their location, but after correcting for the time the light takes to reach them, they also will conclude that the flashes were simultaneous. We can thus define simultaneity as follows:

Two events in a reference frame are simultaneous if light signals from the events reach an observer halfway between the events at the same time.

DEFINITION—SIMULTANEITY

To show that two events that are simultaneous in frame S are not simultaneous in another frame S' moving relative to S, we will use an example introduced by Einstein. A train is moving with speed v past a station platform. We will consider the train to be at rest in S' and the platform to be at rest in S. We have observers A', B', and C' at the front, back, and middle of the train. We now suppose that the train and platform are struck by lightning at the front and back of the train and that the lightning bolts are simultaneous in the frame of the platform S (Figure 39-5). That is, an observer C on the platform halfway between the positions A and B, where the lightning strikes, sees the two flashes at the same time. It is convenient to suppose that the lightning scorches the train and platform so that the events can be easily located. Because C' is in the middle of the train, halfway between the places on the train that are scorched, the events are simultaneous in S' only if C' sees the flashes at the same time. However, the flash from the front of the train is seen by C' before the flash from the back of the

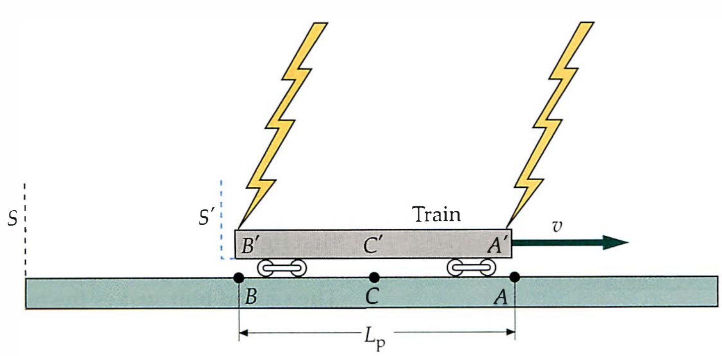


FIGURE 39-5 In frame S attached to the platform, simultaneous lightning bolts strike the ends of a train traveling with speed v. The light from these simultaneous events reaches observer C, standing midway between the events, at the same time. The distance between the bolts is $L_{p,\text{platform}}$.

train. We can understand this by considering the motion of C' as seen in frame S (Figure 39-6). By the time the light from the front flash reaches C', C' has moved some distance toward the front flash and some distance away from the back flash. Thus, the light from the back flash has not yet reached C', as indicated in the figure. Observer C' must therefore conclude that the events are not simultaneous and that the front of the train was struck before the back. Furthermore, all observers in S' on the train will agree with C' when they have corrected for the time it takes the light to reach them.

Figure 39-7 shows the events of the lightning bolts as seen in the reference frame of the train (S'). In this frame the platform is moving, so the distance between the burns on the platform is contracted. The platform is shorter than it is in S, and, since the train is at rest, the train is longer than its contracted length in S. When the lightning bolt strikes the front of the train at A', the front of the train is at point A, and the back of the train has not yet reached point B. Later, when the lightning bolt strikes the back of the train at B', the back has reached point B on the platform.

The time discrepancy of two clocks that are synchronized in frame S as seen in frame S' can be found from the Lorentz transformation equations. Suppose we have clocks at points x_1 and x_2 that are synchronized in S. What are the times t_1 and t_2 on these clocks as observed from frame S' at a time t_0'? From Equation 39-12, we have

$$t_0' = \gamma\left(t_1 - \frac{vx_1}{c^2}\right)$$

and

$$t_0' = \gamma\left(t_2 - \frac{vx_2}{c^2}\right)$$

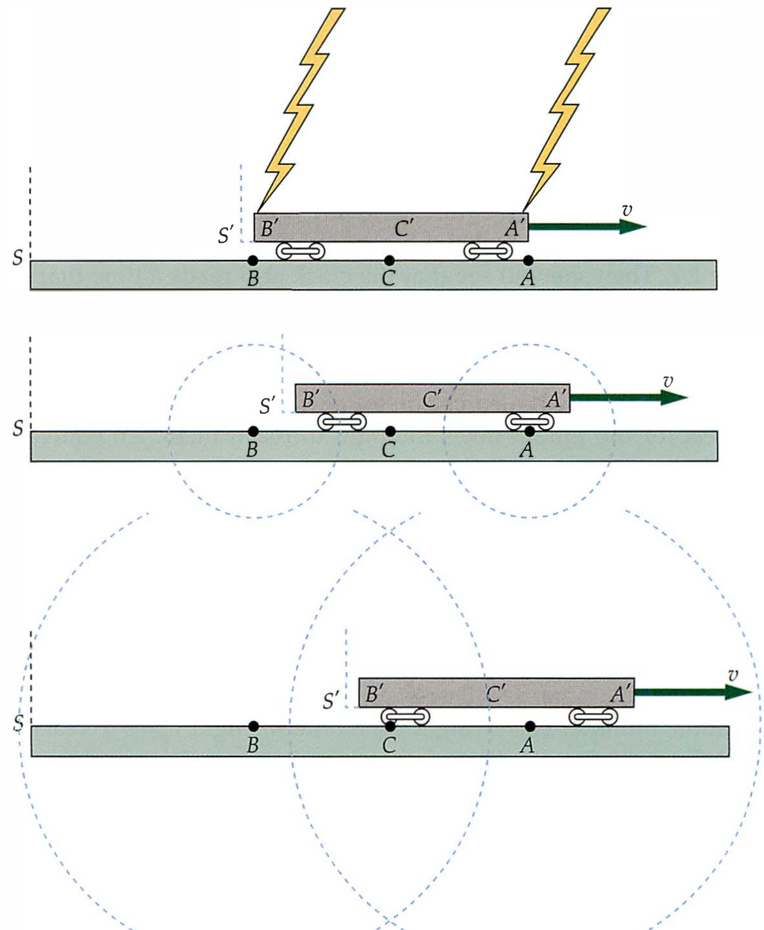


FIGURE 39-6 In frame S attached to the platform, the light from the lightning bolt at the front of the train reaches observer C', standing on the train at its midpoint, before the light from the bolt at the back of the train. Since C' is midway between the events (which occur at the front and rear of the train), these events are not simultaneous for him.

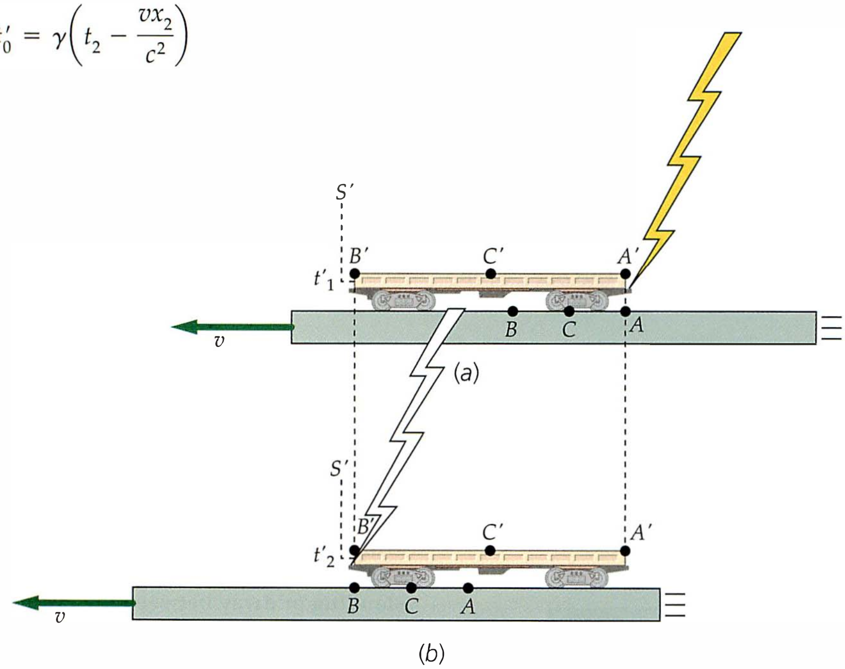


FIGURE 39-7 The lightning bolts of Figure 39-5 as seen in frame S' of the train. In this frame, the distance between A and B on the platform is less than $L_{p,\text{platform}}$, and the proper length of the train $L_{p,\text{train}}$ is longer than $L_{p,\text{platform}}$. The first lightning bolt strikes the front of the train when A' and A are coincident. The second bolt strikes the rear of the train when B' and B are coincident.

Then

$$t_2 - t_1 = \frac{v}{c^2}(x_2 - x_1)$$

Note that the chasing clock (at x_2) leads the other (at x_1) by an amount that is proportional to their proper separation $L_p = x_2 - x_1$.

If two clocks are synchronized in the frame in which they are both at rest, in a frame in which they are moving along the line through both clocks, the chasing clock leads (shows a later time) by an amount

$$\Delta t_S = L_p \frac{v}{c^2} \qquad 39\text{-}17$$

where L_p is the proper distance between the clocks.

CHASING CLOCK SHOWS LATER TIME

A numerical example should help clarify time dilation, clock synchronization, and the internal consistency of these results.

SYNCHRONIZING CLOCKS **EXAMPLE 39-6**

An observer in a spaceship has a flashgun and a mirror, as shown in Figure 39-3. The distance from the gun to the mirror is 15 light-minutes (written 15*c*·min) and the spaceship, at rest in frame *S'*, travels with speed $v = 0.8c$ relative to a very long space platform that is at rest in frame *S*. The platform has two synchronized clocks, one clock at the position x_1 of the spaceship when the observer explodes the flashgun, and the other clock at the position x_2 of the spaceship when the light returns to the gun from the mirror. Find the time intervals between the events (exploding the flashgun and receiving the return flash from the mirror) (*a*) in the frame of the spaceship and (*b*) in the frame of the platform. (*c*) Find the distance traveled by the spaceship and (*d*) the amount by which the clocks on the platform are out of synchronization according to observers on the spaceship.

(*a*) 1. In the spaceship, the light travels from the gun to the mirror and back, a total distance $D = 30\ c\cdot\text{min}$. The time required is D/c:

$$\Delta t' = \frac{D}{c} = \frac{30\ c\cdot\text{min}}{c} = 30\ \text{min}$$

2. Since these events happen at the same place in the spaceship, the time interval is proper time:

$$\Delta t_p = \boxed{30\ \text{min}}$$

(*b*) 1. In frame *S*, the time between the events is longer by the factor γ:

$$\Delta t = \gamma\,\Delta t_p = \gamma(30\ \text{min})$$

2. Calculate γ:

$$\gamma = \frac{1}{\sqrt{1-(v^2/c^2)}} = \frac{1}{\sqrt{1-(0.8)^2}} = \frac{1}{\sqrt{0.36}} = \frac{5}{3}$$

3. Use this value of γ to calculate the time between the events as observed in frame *S*:

$$\Delta t = \gamma\,\Delta t_p = \tfrac{5}{3}(30\ \text{min}) = \boxed{50\ \text{min}}$$

(*c*) In frame *S*, the distance traveled by the spaceship is $v\,\Delta t$:

$$x_2 - x_1 = v\,\Delta t = (0.8c)(50\ \text{min}) = \boxed{40\ c\cdot\text{min}}$$

(*d*) 1. The amount that the clocks on the platform are out of synchronization is related to the proper distance between the clocks L_p:

$$\Delta t_s = L_p \frac{v}{c^2}$$

2. The Part (*c*) result is the proper distance between the clocks on the platform:

$$L_p = x_2 - x_1 = 40\ c\cdot\text{min}$$

so

$$\Delta t_s = L_p \frac{v}{c^2} = (40\ c\cdot\text{min})\frac{(0.8c)}{c^2} = \boxed{32\ \text{min}}$$

REMARKS Observers on the platform would say that the spaceship's clock is running slow because it records a time of only 30 min between the events, whereas the time measured by observers on the platform is 50 min.

Figure 39-8 shows the situation viewed from the spaceship in S'. The platform is traveling past the ship with speed $0.8c$. There is a clock at point x_1, which coincides with the ship when the flashgun is exploded, and another at point x_2, which coincides with the ship when the return flash is received from the mirror. We assume that the clock at x_1 reads 12:00 noon at the time of the light flash. The clocks at x_1 and x_2 are synchronized in S but not in S'. In S', the clock at x_2, which is chasing the one at x_1, leads by 32 min; it would thus read 12:32 to an observer in S'. When the spaceship coincides with x_2, the clock there reads 12:50. The time between the events is therefore 50 min in S. Note that according to observers in S', this clock ticks off 50 min − 32 min = 18 min for a trip that takes 30 min in S'. Thus, observers in S' see this clock run slow by the factor $30/18 = 5/3$.

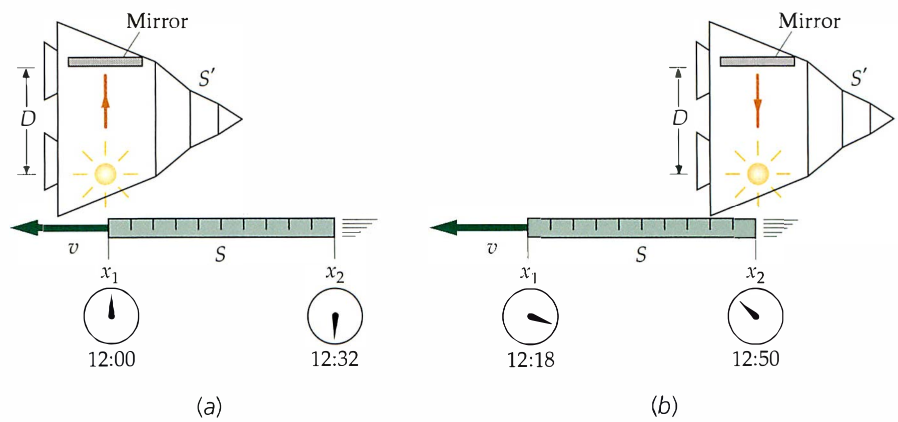


FIGURE 39-8 Clocks on a platform as observed from the spaceship's frame of reference S'. During the time $\Delta t'$ = 30 min it takes for the platform to pass the spaceship, the clocks on the platform run slow and tick off (30 min)/γ = 18 min. But the clocks are unsynchronized, with the chasing clock leading by $L_p v/c^2$, which for this case is 32 min. The time it takes for the spaceship to go from x_1 to x_2, as measured on the platform, is therefore 32 min + 18 min = 50 min.

Every observer in one frame sees the clocks in the other frame run slow. According to observers in S, who measure 50 min for the time interval, the time interval in S' (30 min) is too small, so they see the single clock in S' run too slow by the factor 5/3. According to the observers in S', the observers in S measure a time that is too *long* despite the fact that their clocks run too slow because the clocks in S are out of synchronization. The clocks tick off only 18 min, but the second clock leads the first clock by 32 min, so the time interval is 50 min.

The Twin Paradox

Homer and Ulysses are identical twins. Ulysses travels at high speed to a planet beyond the solar system and returns while Homer remains at home. When they are together again, which twin is older, or are they the same age? The correct answer is that Homer, the twin who stays at home, is older. This problem, with variations, has been the subject of spirited debate for decades, though there are very few who disagree with the answer. The problem appears to be a paradox because of the seemingly symmetric roles played by the twins with the asymmetric result in their aging. The paradox is resolved when the asymmetry of the twins' roles is noted. The relativistic result conflicts with common sense based on our strong but incorrect belief in absolute simultaneity. We will consider a particular case with some numerical magnitudes that, though impractical, make the calculations easy.

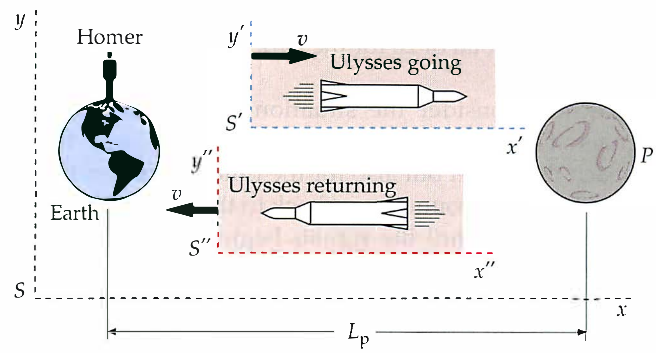


FIGURE 39-9 The twin paradox. The earth and a distant planet are fixed in frame S. Ulysses coasts in frame S' to the planet and then coasts back in frame S''. His twin Homer stays on the earth. When Ulysses returns, he is younger than his twin. The roles played by the twins are not symmetric. Homer remains in one inertial reference frame, but Ulysses must accelerate if he is to return home.

Let planet P and Homer on the earth be at rest in reference frame S a distance L_p apart, as illustrated in Figure 39-9. We neglect the motion of the earth. Reference frames S' and S'' are moving with speed v toward and away from the planet, respectively. Ulysses quickly accelerates to speed v, then coasts in S' until he reaches the planet, where he quickly decelerates to a stop and is momentarily at rest in S. To return, Ulysses quickly accelerates to speed v toward the earth and then coasts in S'' until he reaches the earth, where he quickly decelerates to a stop. We can assume that the acceleration (and deceleration) times are negligible compared with the coasting times. We use the following values for illustration: $L_p = 8$ light-years ($8\ c\cdot\text{y}$) and $v = 0.8c$. Then $\sqrt{1 - (v^2/c^2)} = 3/5$ and $\gamma = 5/3$.

It is easy to analyze the problem from Homer's point of view on the earth. According to Homer's clock, Ulysses coasts in S' for a time $L_p/v = 10$ y and in S'' for an equal time. Thus, Homer is 20 y older when Ulysses returns. The time interval in S' between Ulysses's leaving the earth and his arriving at the planet is shorter because it is proper time. The time it takes to reach the planet by Ulysses's clock is

$$\Delta t' = \frac{\Delta t}{\gamma} = \frac{10\text{ y}}{5/3} = 6\text{ y}$$

Since the same time is required for the return trip, Ulysses will have recorded 12 y for the round trip and will be 8 y younger than Homer upon his return.

From Ulysses's point of view, the distance from the earth to the planet is contracted and is only

$$L' = \frac{L_p}{\gamma} = \frac{8\ c\cdot\text{y}}{5/3} = 4.8\ c\cdot\text{y}$$

At $v = 0.8c$, it takes only 6 y each way.

The real difficulty in this problem is for Ulysses to understand why his twin aged 20 y during his absence. If we consider Ulysses as being at rest and Homer as moving away, Homer's clock should run slow and measure only $3/5(6\text{ y}) = 3.6$ y. Then why shouldn't Homer age only 7.2 y during the round trip? This, of course, is the paradox. The difficulty with the analysis from the point of view of Ulysses is that he does not remain in an inertial frame. What happens while Ulysses is stopping and starting? To investigate this problem in detail, we would need to treat accelerated reference frames, a subject dealt with in the study of general relativity and beyond the scope of this book. However, we can get some insight into the problem by having the twins send regular signals to each other so that they can record the other's age continuously. If they arrange to send a signal once a year, each can determine the age of the other merely by counting the signals received. The arrival frequency of the signals will not be 1 per year because of the Doppler shift. The frequency observed will be given by Equation 39-16*a* and Equation 39-16*b*. Using $v/c = 0.8$ and $v^2/c^2 = 0.64$, we have for the case in which the twins are receding from each other

$$f' = \frac{\sqrt{1 - (v^2/c^2)}}{1 + (v/c)} f_0 = \frac{\sqrt{1 - 0.64}}{1 + 0.8} f_0 = \frac{1}{3} f_0$$

When they are approaching, Equation 39-16*a* gives $f' = 3f_0$.

Consider the situation first from the point of view of Ulysses. During the 6 y it takes him to reach the planet (remember that the distance is contracted in his frame), he receives signals at the rate of $\frac{1}{3}$ signal per year, and so he receives 2 signals. As soon as Ulysses turns around and starts back to the earth, he begins to receive 3 signals per year. In the 6 y it takes him to return he receives 18 signals, giving a total of 20 for the trip. He accordingly expects his twin to have aged 20 years.

We now consider the situation from Homer's point of view. He receives signals at the rate of $\frac{1}{3}$ signal per year not only for the 10 y it takes Ulysses to reach the planet but also for the time it takes for the last signal sent by Ulysses before he turns around to get back to the earth. (He cannot know that Ulysses has turned around until the signals begin reaching him with increased frequency.) Since the planet is 8 light-years away, there is an additional 8 y of receiving signals at the rate of $\frac{1}{3}$ signal per year. During the first 18 y, Homer receives 6 signals. In the final 2 y before Ulysses arrives, Homer receives 6 signals, or 3 per year. (The first signal sent after Ulysses turns around takes 8 y to reach the earth, whereas Ulysses, traveling at $0.8c$, takes 10 y to return and therefore arrives just 2 y after Homer begins to receive signals at the faster rate.) Thus, Homer expects Ulysses to have aged 12 y. In this analysis, the asymmetry of the twins' roles is apparent. When they are together again, both twins agree that the one who has been accelerated will be younger than the one who stayed home.

The predictions of the special theory of relativity concerning the twin paradox have been tested using small particles that can be accelerated to such large speeds that γ is appreciably greater than 1. Unstable particles can be accelerated and trapped in circular orbits in a magnetic field, for example, and their lifetimes can then be compared with those of identical particles at rest. In all such experiments, the accelerated particles live longer on the average than the particles at rest, as predicted. These predictions have also been confirmed by the results of an experiment in which high-precision atomic clocks were flown around the world in commercial airplanes, but the analysis of this experiment is complicated due to the necessity of including gravitational effects treated in the general theory of relativity.

39-5 The Velocity Transformation

We can find how velocities transform from one reference frame to another by differentiating the Lorentz transformation equations. Suppose a particle has velocity $u'_x = dx'/dt'$ in frame S', which is moving to the right with speed v relative to frame S. The particle's velocity in frame S is

$$u_x = \frac{dx}{dt}$$

From the Lorentz transformation equations (Equation 39-9 and Equation 39-10), we have

$$dx = \gamma(dx' + v\,dt')$$

and

$$dt = \gamma\left(dt' + \frac{v\,dx'}{c^2}\right)$$

The velocity in S is thus

$$u_x = \frac{dx}{dt} = \frac{\gamma\,(dx' + v\,dt')}{\gamma\left(dt' + \dfrac{v\,dx'}{c^2}\right)} = \frac{\dfrac{dx'}{dt'} + v}{1 + \dfrac{v}{c^2}\dfrac{dx'}{dt'}} = \frac{u'_x + v}{1 + \dfrac{v\,u'_x}{c^2}}$$

If a particle has components of velocity along the y or z axes, we can use the same relation between dt and dt', with $dy = dy'$ and $dz = dz'$, to obtain

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt' + \dfrac{v\,dx'}{c^2}\right)} = \frac{\dfrac{dy'}{dt'}}{\gamma\left(1 + \dfrac{v}{c^2}\dfrac{dx'}{dt'}\right)} = \frac{u_y'}{\gamma\left(1 + \dfrac{vu_x'}{c^2}\right)}$$

and

$$u_z = \frac{u_z'}{\gamma\left(1 + \dfrac{vu_x'}{c^2}\right)}$$

The complete relativistic velocity transformation is

$$u_x = \frac{u_x' + v}{1 + \dfrac{vu_x'}{c^2}} \qquad \text{39-18a}$$

$$u_y = \frac{u_y'}{\gamma\left(1 + \dfrac{vu_x'}{c^2}\right)} \qquad \text{39-18b}$$

$$u_z = \frac{u_z'}{\gamma\left(1 + \dfrac{vu_x'}{c^2}\right)} \qquad \text{39-18c}$$

RELATIVISTIC VELOCITY TRANSFORMATION

The inverse velocity transformation equations are

$$u_x' = \frac{u_x - v}{1 - \dfrac{vu_x}{c^2}} \qquad \text{39-19a}$$

$$u_y' = \frac{u_y}{\gamma\left(1 - \dfrac{vu_x}{c^2}\right)} \qquad \text{39-19b}$$

$$u_z' = \frac{u_z}{\gamma\left(1 - \dfrac{vu_x}{c^2}\right)} \qquad \text{39-19c}$$

These equations differ from the classical and intuitive result $u_x = u_x' + v$, $u_y = u_y'$, and $u_z = u_z'$ because the denominators in the equations are not equal to 1. When v and u_x' are small compared with the speed of light c, $\gamma \approx 1$ and $vu_x'/c^2 << 1$. Then the relativistic and classical expressions are the same.

RELATIVE VELOCITY AT NONRELATIVISTIC SPEEDS — **EXAMPLE 39-7**

A supersonic plane moves away from you along the x axis with speed 1000 m/s (about 3 times the speed of sound) relative to you. A second plane moves along the x axis away from you, and away from the first plane, at speed 500 m/s relative to the first plane. How fast is the second plane moving relative to you?

PICTURE THE PROBLEM These speeds are so small compared with c that we expect the classical equations for combining velocities to be accurate. We show this by calculating the correction term in the denominator of Equation 39-18a. Let frame S be your rest frame and frame S' be moving with velocity $v = 1000$ m/s. The first plane is then at rest in frame S', and the second plane has velocity $u_x' = 500$ m/s in S'.

1. Let S and S' be the reference frames of you and the first plane, respectively. Also, let u_x and u_x' be the velocities of the second plane relative to S and S', respectively. Equation 39-18a can be used to find u_x. The velocity of the second plane relative to you is v:

$$u_x = \frac{u_x' + v}{1 + \dfrac{vu_x'}{c^2}}$$

2. If the correction term in the denominator is neglgible, Equation 39-18a gives the classical formula for combining velocities. Calculate the value of this correction term:

$$\frac{vu_x'}{c^2} = \frac{(1000)(500)}{(3 \times 10^8)^2} \approx 5.6 \times 10^{-12}$$

3. This correction term is so small that the classical and relativistic results are essentially the same:

$$u_x \approx u_x' + v$$
$$= 500 \text{ m/s} + 1000 \text{ m/s} = \boxed{1500 \text{ m/s}}$$

Relative Velocity at Relativistic Speeds — EXAMPLE 39-8

Work Example 39-7 if the first plane moves with speed $v = 0.8c$ relative to you and the second plane moves with the same speed $0.8c$ relative to the first plane.

PICTURE THE PROBLEM These speeds are not small compared with c, so we use the relativistic expression (Equation 39-18a). We again assume that you are at rest in frame S and the first plane is at rest in frame S' that is moving at $v = 0.8c$ relative to you. The velocity of the second plane in S' is $u_x' = 0.8c$.

Use Equation 39-18a to calculate the speed of the second plane relative to you:

$$u_x = \frac{u_x' + v}{1 + \dfrac{vu_x'}{c^2}} = \frac{0.8c + 0.8c}{1 + \dfrac{(0.8c)(0.8c)}{c^2}} = \frac{1.6c}{1.64} = \boxed{0.98c}$$

The result in Example 39-8 is quite different from the classically expected result of $0.8c + 0.8c = 1.6c$. In fact, it can be shown from Equations 39-18 that if the speed of an object is less than c in one frame, it is less than c in all other frames moving relative to that frame with a speed less than c. (See Problem 23.) We will see in Section 39-7 that it takes an infinite amount of energy to accelerate a particle to the speed of light. The speed of light c is thus an upper, unattainable limit for the speed of a particle with mass. (There are massless particles, such as photons, that always move at the speed of light.)

Relative Speed of a Photon — EXAMPLE 39-9

A photon moves along the x axis in frame S', with speed $u_x' = c$. What is its speed in frame S?

The speed in S is given by Equation 39-18a:

$$u_x = \frac{u_x' + v}{1 + \dfrac{vu_x'}{c^2}} = \frac{c + v}{1 + \dfrac{vc}{c^2}} = \frac{c + v}{1 + \dfrac{v}{c}} = \frac{c + v}{\dfrac{1}{c}(c + v)} = \boxed{c}$$

REMARKS The speed in both frames is c, independent of v. This is in accord with Einstein's postulates.

Rockets Passing in Opposite Directions **EXAMPLE 39-10**

Two spaceships, each 100 m long when measured at rest, travel toward each other with speeds of 0.85c relative to the earth. (a) How long is each spaceship as measured by someone on the earth? (b) How fast is each spaceship traveling as measured by an observer on one of the spaceships? (c) How long is one spaceship when measured by an observer on one of the spaceships? (d) At time $t = 0$ on the earth, the front ends of the ships are together as they just begin to pass each other. At what time on the earth are their back ends together?

PICTURE THE PROBLEM (a) The length of each spaceship as measured on the earth is the contracted length $\sqrt{1-(v_1^2/c^2)}\ L_p$ (Equation 39-14), where v_1 is the speed of either spaceship. To solve Part (b), let the earth be in frame S, and the spaceship on the left be in frame S' moving with velocity $v = 0.85c$ relative to S. Then the spaceship on the right moves with velocity $u_x = -0.85c$, as shown in Figure 39-10. (c) The length of one spaceship as seen by the other is $\sqrt{1-(v_2^2/c^2)}\ L_p$, where v_2 is the speed of one spaceship relative to the other.

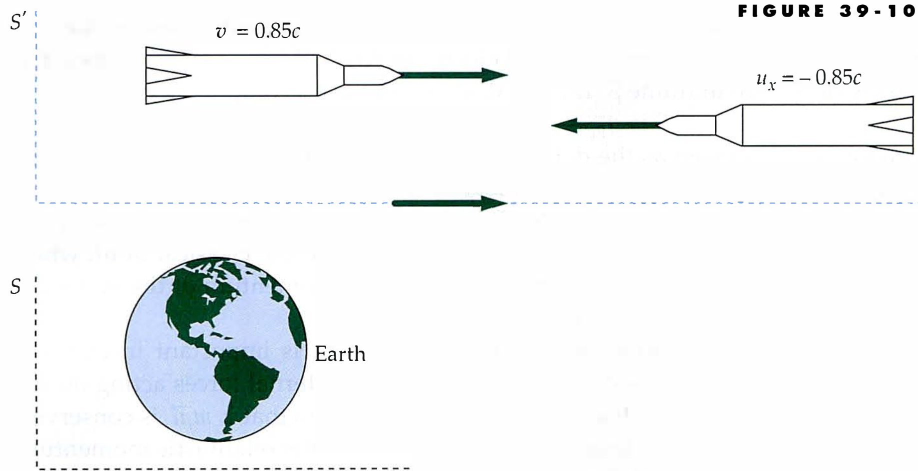


FIGURE 39-10

(a) The length of each spaceship in the earth's frame is the proper length divided by γ:

$$L = \sqrt{1-\frac{v_1^2}{c^2}}\,L_p = \sqrt{1-\frac{(0.85c)^2}{c^2}}\,(100\text{ m}) = \boxed{52.7\text{ m}}$$

(b) Use the velocity transformation formula (Equation 39-19a) to find the velocity u'_x of the spaceship on the right as seen in frame S':

$$u'_x = \frac{u_x - v}{1-\dfrac{vu_x}{c^2}} = \frac{-0.85c - 0.85c}{1-\dfrac{(0.85c)(-0.85c)}{c^2}} = \frac{-1.70c}{1.7225} = \boxed{-0.987c}$$

(c) In the frame of the left spaceship, the right spaceship is moving with speed $v_2 = |u'_x| = 0.987c$. Use this to calculate the contracted length of the spaceship on the right:

$$L = \sqrt{1-\frac{v_1^2}{c^2}}\,L_p = \sqrt{1-\frac{(0.987c)^2}{c^2}}\,(100\text{ m}) = \boxed{16.1\text{ m}}$$

(d) If the front ends of the spaceships are together at $t = 0$ on the earth, their back ends will be together after the time it takes either spaceship to move the length of the spaceship in the earth's frame:

$$t = \frac{L}{v_1} = \frac{52.7\text{ m}}{0.85c} = \frac{52.7\text{ m}}{(0.85)(3\times10^8\text{ m/s})} = \boxed{2.07\times10^{-7}\text{ s}}$$

39-6 Relativistic Momentum

We have seen in previous sections that Einstein's postulates require important modifications in our ideas of simultaneity and in our measurements of time and length. Einstein's postulates also require modifications in our concepts of mass, momentum, and energy. In classical mechanics, the momentum of a particle is defined as the product of its mass and its velocity, $m\vec{u}$, where $\vec{u}$ is the velocity. In an isolated system of particles, with no net force acting on the system, the total momentum of the system remains constant.

We can see from a simple thought experiment that the quantity $\Sigma m_i\vec{u}_i$ is not conserved in an isolated system. We consider two observers: observer A in reference frame S and observer B in frame S', which is moving to the right in the x direction with speed v with respect to frame S. Each has a ball of mass m. The two balls are identical when compared at rest. One observer throws his ball up with a speed u_0 relative to him and the other throws his ball down with a speed u_0 relative to him, so that each ball travels a distance L, makes an elastic collision with the other ball, and returns. Figure 39-11 shows how the collision looks in each reference frame. Classically, each ball has vertical momentum of magnitude mu_0. Since the vertical components of the momenta are equal and opposite, the total vertical component of momentum is zero before the collision. The collision merely reverses the momentum of each ball, so the total vertical momentum is zero after the collision.

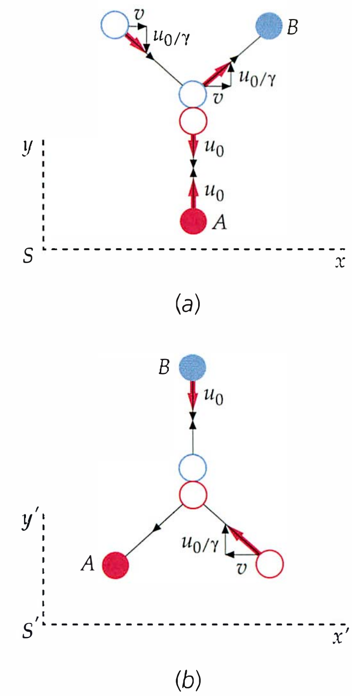


FIGURE 39-11 *(a)* Elastic collision of two identical balls as seen in frame S. The vertical component of the velocity of ball B is u_0/γ in S if it is u_0 in S'. *(b)* The same collision as seen in S'. In this frame, ball A has a vertical component of velocity equal to u_0/γ.

Relativistically, however, the vertical components of the velocities of the two balls as seen by either observer are not equal and opposite. Thus, when they are reversed by the collision, classical momentum is not conserved. Consider the collision as seen by A in frame S. The velocity of his ball is $u_{Ay} = +u_0$. Since the velocity of B's ball in frame S' is $u'_{Bx} = 0$, $u'_{By} = -u_0$, the y component of the velocity of B's ball in frame S is $u_{By} = -u_0/\gamma$ (Equation 39-18*b*). Thus, if the classical expression $m\vec{u}$ is taken as the definition of momentum, the vertical components of momentum of the two balls are not equal and opposite as seen by observer A. Since the balls are reversed by the collision, classical momentum is not conserved. Of course, the same result is observed by B. In the classical limit, when u is much less than c, γ is approximately 1, and the momentum of the system is conserved as seen by either observer.

The reason that the total momentum of a system is important in classical mechanics is that it is conserved when there are no external forces acting on the system, as is the case in collisions. But we have just seen that $\Sigma m_i\vec{u}_i$ is conserved only in the approximation that $u << c$. We will define the relativistic momentum $\vec{p}$ of a particle to have the following properties:

1. In collisions, $\vec{p}$ is conserved.
2. As u/c approaches zero, $\vec{p}$ approaches $m\vec{u}$.

We will show that the quantity

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1 - \dfrac{u^2}{c^2}}} \tag{39-20}$$

RELATIVISTIC MOMENTUM

is conserved in the elastic collision shown in Figure 39-11. Since this quantity also approaches $m\vec{u}$ as u/c approaches zero, we take this equation for the definition of the **relativistic momentum** of a particle.

One interpretation of Equation 39-20 is that the mass of an object increases with speed. Then the quantity $m_{\text{rel}} = m/\sqrt{1 - (u^2/c^2)}$ is called the *relativistic mass*. The relativistic mass of a particle when it is at rest in some reference frame is then called its *rest mass m*. In this chapter, we will treat the terms mass and rest mass as synonymous, and both terms will be labeled m.

Illustration of Conservation of the Relativistic Momentum

We will compute the y component of the relativistic momentum of each particle in the reference frame S for the collision of Figure 39-11 and show that the y component of the total relativistic momentum is zero. The speed of ball A in S is u_0, so the y component of its relativistic momentum is

$$p_{Ay} = \frac{mu_0}{\sqrt{1 - (u_0^2/c^2)}}$$

The speed of ball B in S is more complicated. Its x component is v and its y component is $-u_0/\gamma$. Thus,

$$u_B^2 = u_{Bx}^2 + u_{By}^2 = v^2 + \left[-u_0\sqrt{1 - (v^2/c^2)}\right]^2 = v^2 + u_0^2 - \frac{u_0^2 v^2}{c^2}$$

Using this result to compute $\sqrt{1 - (u_B^2/c^2)}$, we obtain

$$1 - \frac{u_B^2}{c^2} = 1 - \frac{v^2}{c^2} - \frac{u_0^2}{c^2} + \frac{u_0^2 v^2}{c^4} = \left(1 - \frac{v^2}{c^2}\right)\left(1 - \frac{u_0^2}{c^2}\right)$$

and

$$\sqrt{1 - (u_B^2/c^2)} = \sqrt{1 - (v^2/c^2)}\,\sqrt{1 - (u_0^2/c^2)} = (1/\gamma)\,\sqrt{1 - (u_0^2/c^2)}$$

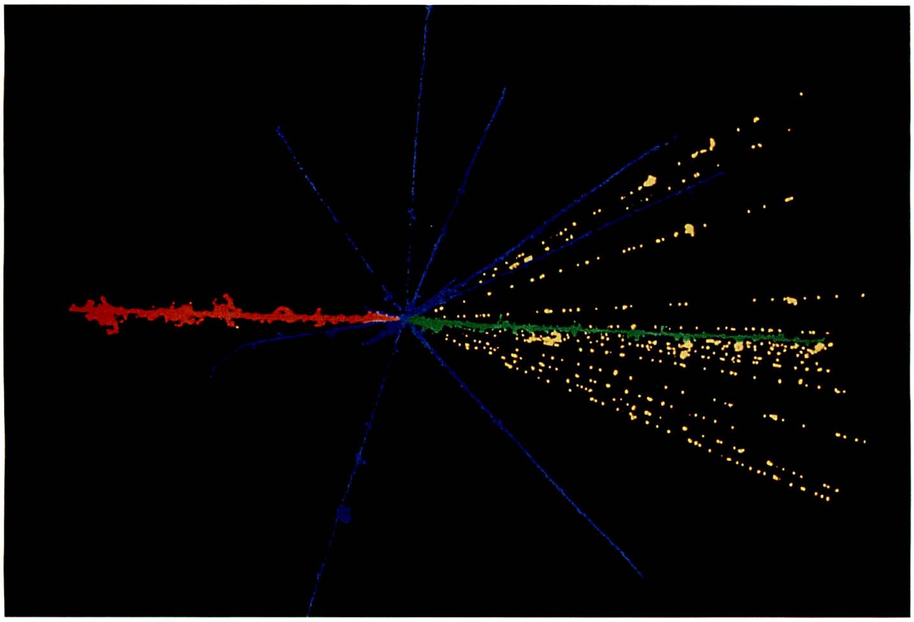
The y component of the relativistic momentum of ball B as seen in S is therefore

$$p_{By} = \frac{mu_{By}}{\sqrt{1 - (u_B^2/c^2)}} = \frac{-mu_0/\gamma}{(1/\gamma)\,\sqrt{1 - (u_0^2/c^2)}} = \frac{-mu_0}{\sqrt{1 - (u_0^2/c^2)}}$$

Since $p_{By} = -p_{Ay}$, the y component of the total momentum of the two balls is zero. If the speed of each ball is reversed by the collision, the total momentum will remain zero and momentum will be conserved.

39-7 Relativistic Energy

In classical mechanics, the work done by the net force acting on a particle equals the change in the kinetic energy of the particle. In relativistic mechanics, we equate the net force to the rate of change of the relativistic momentum. The work done by the net force can then be calculated and set equal to the change in kinetic energy.



The creation of elementary particles demonstrates the conversion of kinetic energy to rest energy. In this 1950 photograph of a cosmic ray shower, a high-energy sulfur nucleus (red) collides with a nucleus in a photographic emulsion and produces a spray of particles, including a fluorine nucleus (green), other nuclear fragments (blue), and approximately 16 pions (yellow).

As in classical mechanics, we will define kinetic energy as the work done by the net force in accelerating a particle from rest to some final velocity u_f. Considering one dimension only, we have

$$K = \int_{u=0}^{u=u_f} F_{net}\, ds = \int_0^{u_f} \frac{dp}{dt}\, ds = \int_0^{u_f} u dp = \int_0^{u_f} u\, d\left(\frac{mu}{\sqrt{1-(u^2/c^2)}}\right) \qquad \text{39-21}$$

where we have used $u = ds/dt$. It is left as a problem (Problem 37) for you to show that

$$d\left(\frac{mu}{\sqrt{1-(u^2/c^2)}}\right) = m\left(1-\frac{u^2}{c^2}\right)^{-3/2} du$$

If we substitute this expression into the integrand in Equation 39-21, we obtain

$$K = \int_0^{u_f} ud\left(\frac{mu}{\sqrt{1-(u^2/c^2)}}\right) = \int_0^{u_f} m\left(1-\frac{u^2}{c^2}\right)^{-3/2} u\, du$$
$$= mc^2\left(\frac{1}{\sqrt{1-(u_f^2/c^2)}} - 1\right)$$

or

$$K = \frac{mc^2}{\sqrt{1-(u^2/c^2)}} - mc^2 \qquad \text{39-22}$$

RELATIVISTIC KINETIC ENERGY

(In this expression the final speed u_f is arbitrary, so the subscript f is not needed.)

The expression for kinetic energy consists of two terms. The first term depends on the speed of the particle. The second, mc^2, is independent of the speed. The quantity mc^2 is called the **rest energy** E_0 of the particle. The rest energy is the product of the mass and c^2:

$$E_0 = mc^2 \qquad \text{39-23}$$

REST ENERGY

The total **relativistic energy** E is then defined to be the sum of the kinetic energy and the rest energy:

$$E = K + mc^2 = \frac{mc^2}{\sqrt{1-(u^2/c^2)}} \qquad \text{39-24}$$

RELATIVISTIC ENERGY

Thus, the work done by an unbalanced force increases the energy from the rest energy mc^2 to the final energy $mc^2/\sqrt{1-(u^2/c^2)} = m_{rel}c^2$, where $m_{rel} = m/\sqrt{1-(u^2/c^2)}$ is the relativistic mass. We can obtain a useful expression for the velocity of a particle by multiplying Equation 39-20 for the relativistic momentum by c^2 and comparing the result with Equation 39-24 for the relativistic energy. We have

$$pc^2 = \frac{mc^2u}{\sqrt{1-(u^2/c^2)}} = Eu$$

or

$$\frac{u}{c} = \frac{pc}{E} \qquad \text{39-25}$$

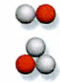
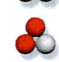
Energies in atomic and nuclear physics are usually expressed in units of electron volts (eV) or mega-electron volts (MeV):

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$$

A convenient unit for the masses of atomic particles is eV/c^2 or MeV/c^2, which is the rest energy of the particle divided by c^2. The rest energies of some elementary particles and light nuclei are given in Table 39-1.

TABLE 39-1

Rest Energies of Some Elementary Particles and Light Nuclei

Particle	Symbol	Rest energy, MeV	
Photon	γ	0	
Electron (positron)	e or e^- (e^+)	0.5110	
Muon	$\mu^\pm$	105.7	
Pion	π^0	135	
	$\pi^\pm$	139.6	
Proton	p	938.280	
Neutron	n	939.573	
Deuteron	^{2}H or d	1875.628	
Triton	^{3}H or t	2808.944	
Helium-3	^{3}He	2808.41	
Alpha particle	^{4}He or α	3727.409	

Total Energy, Kinetic Energy, and Momentum **EXAMPLE 39-11**

An electron (rest energy 0.511 MeV) moves with speed $u = 0.8c$. Find (*a*) its total energy, (*b*) its kinetic energy, and (*c*) the magnitude of its momentum.

(*a*) The total energy is given by Equation 39-24:

$$E = \frac{mc^2}{\sqrt{1-(u^2/c^2)}} = \frac{0.511 \text{ MeV}}{\sqrt{1-0.64}} = \frac{0.511 \text{ MeV}}{0.6} = \boxed{0.852 \text{ MeV}}$$

(*b*) The kinetic energy is the total energy minus the rest energy:

$$K = E - mc^2 = 0.852 \text{ MeV} - 0.511 \text{ MeV} = \boxed{0.341 \text{MeV}}$$

(*c*) The magnitude of the momentum is found from Equation 39-20. We can simplify by multiplying both numerator and denominator by c^2 and using the Part (*a*) result:

$$p = \frac{mu}{\sqrt{1-(u^2/c^2)}}$$

$$= \frac{mc^2}{\sqrt{1-(u^2/c^2)}}\frac{u}{c^2} = (0.852 \text{ MeV})\frac{0.8c}{c^2} = \boxed{0.681 \text{ MeV}/c}$$

REMARKS The technique used to solve Part (*c*) (multiplying numerator and denominator by c^2) is equivalent to using Equation 39-25.

The expression for kinetic energy given by Equation 39-22 does not look much like the classical expression $\frac{1}{2}mu^2$. However, when u is much less than c, we can approximate $1/\sqrt{1-(u^2/c^2)}$ using the binomial expansion

$$(1+x)^n = 1 + nx + n(n-1)\frac{x^2}{2} + \cdots \approx 1 + nx \qquad \text{39-26}$$

Then

$$\frac{1}{\sqrt{1-(u^2/c^2)}} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2}$$

From this result, when u is much less than c, the expression for relativistic kinetic energy becomes

$$K = mc^2\left[\frac{1}{\sqrt{1-(u^2/c^2)}} - 1\right] \approx mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} - 1\right) = \frac{1}{2}mu^2$$

Thus, at low speeds, the relativistic expression is the same as the classical expression.

We note from Equation 39-24 that as the speed u approaches the speed of light c, the energy of the particle becomes very large because $1/\sqrt{1-(u^2/c^2)}$ becomes very large. At $u = c$, the energy becomes infinite. For u greater than c, $\sqrt{1-(u^2/c^2)}$ is the square root of a negative number and is therefore imaginary. A simple interpretation of the result that it takes an infinite amount of energy to accelerate a particle to the speed of light is that no particle that is ever at rest in any inertial reference frame can travel as fast or faster than the speed of light c. As we noted in Example 39-8, if the speed of a particle is less than c in one reference frame, it is less than c in all other reference frames moving relative to that frame at speeds less than c.

In practical applications, the momentum or energy of a particle is often known rather than the speed. Equation 39-20 for the relativistic momentum and Equation 39-24 for the relativistic energy can be combined to eliminate the speed u. The result is

$$E^2 = p^2c^2 + (mc^2)^2 \qquad \text{39-27}$$

RELATION FOR TOTAL ENERGY, MOMENTUM, AND REST ENERGY

This useful equation can be conveniently remembered from the right triangle shown in Figure 39-12. If the energy of a particle is much greater than its rest energy mc^2, the second term on the right side of Equation 39-27 can be neglected, giving the useful approximation

$$E \approx pc, \qquad \text{for } E >> mc^2 \qquad \text{39-28}$$

Equation 39-28 is an exact relation between energy and momentum for particles with no mass, such as photons.

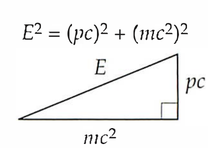


FIGURE 39-12 Right triangle to remember Equation 39-27.

EXERCISE A proton (mass 938 MeV/c^2) has a total energy of 1400 MeV. Find (*a*) $1/\sqrt{1-(u^2/c^2)}$, (*b*) the momentum of the proton, and (*c*) the speed u of the proton. (*Answer* (*a*) 1.49, (*b*) $p = 1.04 \times 10^3$ MeV/c, and (*c*) $u = 0.74c$)

Mass and Energy

Einstein considered Equation 39-23 relating the energy of a particle to its mass to be the most significant result of the theory of relativity. Energy and inertia, which were formerly two distinct concepts, are related through this famous equation. As discussed in Chapter 7, the conversion of rest energy to kinetic energy with a corresponding decrease in mass is a common occurrence in radioactive decay and nuclear reactions, including nuclear fission and nuclear fusion. We illustrated this in Section 7-3 with the deuteron, whose mass is 2.22 MeV/c^2 less than the mass of its parts, a proton and a neutron. When a neutron and a proton combine to form a deuteron, 2.22 MeV of energy is released. The breaking up of a deuteron into a neutron and a proton requires 2.22 MeV of energy input. The proton and the neutron are thus bound together in a deuteron by a binding energy of 2.22 MeV. Any stable composite particle, such as a deuteron or a helium nucleus (2 neutrons plus 2 protons), that is made up of other particles has a mass and rest energy that are less than the sum of the masses and rest energies of its parts. The difference in rest energy is the binding energy of the composite particle. The binding energies of atoms and molecules are of the order of a few electron volts, which leads to a negligible difference in mass between the composite particle and its parts. The binding energies of nuclei are of the order of several MeV, which leads to a noticeable difference in mass. Some very heavy nuclei, such as radium, are radioactive and decay into a lighter nucleus plus an alpha particle. In this case, the original nucleus has a rest energy greater than that of the decay particles. The excess energy appears as the kinetic energy of the decay products.

To further illustrate the interrelation of mass and energy, we consider a perfectly inelastic collision of two particles. Classically, kinetic energy is lost in such a collision. Relativistically, this loss in kinetic energy shows up as an increase in rest energy of the system; that is, the total energy of the system is conserved. Consider a particle of mass m_1 moving with initial speed u_1 that collides with a particle of mass m_2 moving with initial speed u_2. The particles collide and stick together, forming a particle of mass M that moves with speed u_f, as shown in Figure 39-13. The initial total energy of particle 1 is

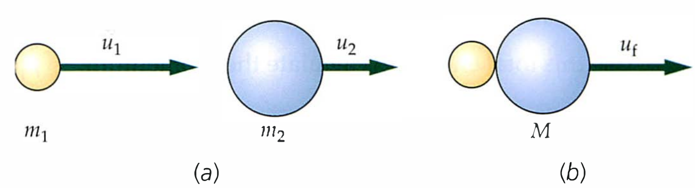


FIGURE 39-13 A perfectly inelastic collision between two particles. One particle of mass m_1 collides with another particle of mass m_2. After the collision, the particles stick together, forming a composite particle of mass M that moves with speed u_f so that relativistic momentum is conserved. Kinetic energy is lost in this process. If we assume that the total energy is conserved, the loss in kinetic energy must equal c^2 times the increase in the mass of the system.

$$E_1 = K_1 + m_1c^2$$

where K_1 is its initial kinetic energy. Similarly the initial total energy of particle 2 is

$$E_2 = K_2 + m_2c^2$$

The total initial energy of the system is

$$E_i = E_1 + E_2 = K_1 + m_1c^2 + K_2 + m_2c^2 = K_i + M_ic^2$$

where $K_i = K_1 + K_2$ and $M_i = m_1 + m_2$ are the initial kinetic energy and initial mass of the system. The final total energy of the system is

$$E_f = K_f + M_fc^2$$

If we set the final total energy equal to the initial total energy, we obtain

$$K_f + M_fc^2 = K_i + M_ic^2$$

Rearranging gives $K_f - K_i = -(M_f - M_i)c^2$, which can be expressed

$$\Delta K + (\Delta M)c^2 = 0 \qquad 39\text{-}29$$

where $\Delta M = M_f - M_i$ is the change in mass of the system.

Totally Inelastic Collision **EXAMPLE 39-12**

A particle of mass 2 MeV/c^2 and kinetic energy 3 MeV collides with a stationary particle of mass 4 MeV/c^2. After the collision, the two particles stick together. Find (*a*) the initial momentum of the system, (*b*) the final velocity of the two-particle system, and (*c*) the mass of the two-particle system.

PICTURE THE PROBLEM (*a*) The initial momentum of the system is the initial momentum of the incoming particle, which can be found from the total energy of the particle. (*b*) The final velocity of the system can be found from its total energy and momentum using $u/c = pc/E$ (Equation 39-25). The energy is found from conservation of energy, and the momentum from conservation of momentum. (*c*) Since the final energy and momentum are known, the final mass can be found from $E^2 = p^2c^2 + (Mc^2)^2$.

(*a*) 1. The initial momentum of the system is the initial momentum of the incoming particle. The momentum of a particle is related to its energy and mass (Equation 39-27):

$$E_1^2 = p_1^2c^2 + (m_1c^2)^2$$
$$p_1c = \sqrt{E_1^2 - (m_1c^2)^2}$$

2. The total energy of the moving particle is the sum of its kinetic energy and its rest energy:

$$E_1 = 3\text{ MeV} + 2\text{ MeV} = 5\text{ MeV}$$

3. Use this total energy to calculate the momentum:

$$p_1c = \sqrt{E_1^2 - (m_1c^2)^2} = \sqrt{(5\text{ MeV})^2 - (2\text{ MeV})^2} = \sqrt{21}\text{ MeV}$$
$$p_1 = \boxed{4.58\text{ MeV}/c}$$

(*b*) 1. We can find the final velocity of the system from its total energy E_f and its momentum p_f using Equation 39-25:

$$\frac{u_f}{c} = \frac{p_fc}{E_f}$$

2. By the conservation of total energy, the final energy of the system equals the initial total energy of the two particles:

$$E_f = E_i = E_1 + E_2 = 5\text{ MeV} + 4\text{ MeV} = 9\text{ MeV}$$

3. By the conservation of momentum, the final momentum of the two-particle system equals the initial momentum:

$$p_f = 4.58\text{ MeV}/c$$

4. Calculate the velocity of the two-particle system from its total energy and momentum using $u/c = pc/E$:

$$\frac{u_f}{c} = \frac{p_fc}{E_f} = \frac{4.58\text{ MeV}}{9\text{ MeV}} = 0.509$$
$$u_f = \boxed{0.509c}$$

(*c*) We can find the mass M_f of the final two-particle system from Equation 39-27 using $pc = 4.58$ MeV and $E = 9$ MeV:

$$E_f^2 = (p_fc)^2 + (M_fc^2)^2$$
$$(9\text{ MeV})^2 = (4.58\text{ MeV})^2 + (M_fc^2)^2$$
$$M_f = \boxed{7.75\text{ MeV}/c^2}$$

REMARKS Note that the mass of the system increased from 6 MeV/c^2 to 7.75 MeV/c^2. This increase times c^2 equals the loss in kinetic energy of the system, as you will show in the following exercise.

EXERCISE (*a*) Find the final kinetic energy of the two-particle system in Example 39-12. (*b*) Find the loss in kinetic energy, K_{loss}, in the collision. (*c*) Show that $K_{loss} = (\Delta M)c^2$, where ΔM is the change in mass of the system. [*Answer* (*a*) $K_f = E_f - M_fc^2 = 9\text{ MeV} - 7.75\text{ MeV} = 1.25\text{ MeV}$, (*b*) $K_{loss} = K_i - K_f = 3\text{ MeV} - 1.25\text{ MeV} = 1.75\text{ MeV}$, and (*c*) $(\Delta M)c^2 = (M_f - M_i)c^2 = 7.75\text{ MeV} - (2\text{ MeV} + 4\text{ MeV}) = 1.75\text{ MeV} = K_{loss}$]

MOMENTUM AND TOTAL-ENERGY CONSERVATION **EXAMPLE 39-13**

A 1×10^6-kg rocket has 1×10^3 kg of fuel on board. The rocket is parked in space when it suddenly becomes necessary to accelerate. The rocket engines ignite, and the 1×10^3 kg of fuel are consumed. The exhaust (spent fuel) is ejected during a very short time interval at a speed of $c/2$ relative to S—the inertial reference frame in which the rocket is initially at rest. (*a*) Calculate the change in the mass of the rocket-fuel system. (*b*) Calculate the final speed of the rocket u_R relative to S. (*c*) Again, calculate the final speed of the rocket relative to S, this time using classical (newtonian) mechanics.

PICTURE THE PROBLEM The speed of the rocket and the change in the mass of the system can be calculated via conservation of momentum and conservation of energy. In reference frame S, the total momentum of the rocket plus fuel is zero. After the burn, the magnitude of the momentum of the rocket equals that of the ejected fuel. Let $m_R = 1 \times 10^6$ kg be the mass of the rocket, not including the mass of the fuel, let $m_{F,i} = 1 \times 10^3$ kg be the mass of the fuel *before* the burn, and let $m_{F,f}$ be the mass of the fuel *after* the burn. The mass of the rocket, m_R, remains fixed, but during the burn the mass of the fuel decreases. (The fuel has less chemical energy after the burn, and so has less mass as well.)

(*a*) 1. The magnitudes of the momentum of the rocket and the momentum of the ejected fuel are equal. For the reasons stated above, the mass of the rocket, not including the 1×10^3 kg of fuel, does not change during the burn:

$$p_R = p_F$$

$$\frac{m_R u_R}{\sqrt{1 - (u_R^2/c^2)}} = \frac{m_{F,f} u_F}{\sqrt{1 - (u_F^2/c^2)}} = p$$

where

$p = p_R = p_F$, $m_R = 1 \times 10^6$ kg, $u_F = 0.5c$, and u_R is the final speed of the rocket.

2. The total energy of the system does not change:

$$E_f = E_i$$

3. The initial energy is the rest energy of the rocket and fuel before the burn. The final energy is the energy of the rocket plus energy of the fuel. The energy of each is related to its momentum by Equation 39-27:

$$E_i = m_R c^2 + m_{F,i} c^2 = (m_R + m_{F,i})c^2$$

$$E_{R,f}^2 = p^2c^2 + (m_R c^2)^2$$

$$E_{F,f}^2 = p^2c^2 + (m_{F,f} c^2)^2$$

so

$$E_f = E_{R,f} + E_{F,f}$$

$$E_f = \sqrt{p^2c^2 + (m_R c^2)^2} + \sqrt{p^2c^2 + (m_{F,f} c^2)^2}$$

4. Equate the initial and final energies:

$$\sqrt{p^2c^2 + (m_R c^2)^2} + \sqrt{p^2c^2 + (m_{F,f} c^2)^2} = (m_R + m_{F,i})c^2$$

5. The step 4 result and the step 1 result, $p = \dfrac{m_{F,f} u_F}{\sqrt{1 - (u_F^2/c^2)}}$, constitute two simultaneous equations with unknowns p and $m_{F,f}$. Solving for $m_{F,f}$ gives:

$$m_{F,f} = 866 \text{ kg}$$

so

$$m_{\text{loss}} = m_{F,i} - m_{F,f} = 1000 \text{ kg} - 866 \text{ kg} = \boxed{134 \text{ kg}}$$

(*b*) 1. To solve for u_R, we use Equation 39-25:

$$\frac{u_R}{c} = \frac{pc}{E_{R,f}}$$

2. To solve for p, we substitute the value for $m_{F,f}$ into the Part (*a*), step 1 result:

$$p = \frac{m_{F,f} u_F}{\sqrt{1 - (u_F^2/c^2)}} = \frac{(866 \text{ kg})\frac{1}{2}c}{\sqrt{1 - \frac{1}{4}}} = (5.00 \times 10^2 \text{ kg})c$$

3. We use the value for p to solve for $E_{R,f}$:

$$E^2_{R,f} = p^2c^2 + (m_Rc^2)^2 = (5.00 \times 10^2\ \text{kg})^2c^4 + (10^6\ \text{kg})^2c^4 = (1.00 \times 10^{12}\ \text{kg}^2)c^4$$

so

$$E_{R,f} = (1.00 \times 10^6\ \text{kg})c^2$$

4. Using our Part (*b*), step 1 result, we solve for u_R:

$$u_R = \frac{pc^2}{E_{R,f}} = \frac{(5.00 \times 10^2\ \text{kg})c^3}{(1.00 \times 10^6\ \text{kg})c^2} = \boxed{5.00 \times 10^{-4}c = 1.50 \times 10^5\ \text{m/s}}$$

(*c*) Equate the magnitude of the classical expressions for the momentum of the rocket and burned fuel and solve for u_R:

$$m_Ru_R = m_Fu_F$$

$$u_R = \frac{m_F}{m_R}u_F = \frac{10^3\ \text{kg}}{10^6\ \text{kg}}0.5c = \boxed{1.5 \times 10^5\ \text{m/s}}$$

REMARK If carried out to five figures, the relativistic calculation gives $u_R = 4.9994 \times 10^4\ c$ for the final speed of the rocket. However, the classical calculation gives $u_R = 5.0000 \times 10^4\ c$. These two values differ by less than one part in 8000.

MASTER the CONCEPT WEB

EXERCISE If the matter being ejected were a 1×10^3-kg rigid block launched by a spring with one end attached to the rocket, would the rest mass of the block change or would the rest mass of the spring change? (*Answer* Only the rest mass of the spring would change.)

39-8 General Relativity

The generalization of the theory of relativity to noninertial reference frames by Einstein in 1916 is known as the general theory of relativity. It is much more difficult mathematically than the special theory of relativity, and there are fewer situations in which it can be tested. Nevertheless, its importance calls for a brief qualitative discussion.

The basis of the general theory of relativity is the **principle of equivalence:**

A homogeneous gravitational field is completely equivalent to a uniformly accelerated reference frame.

PRINCIPLE OF EQUIVALENCE

This principle arises in Newtonian mechanics because of the apparent identity of gravitational mass and inertial mass. In a uniform gravitational field, all objects fall with the same acceleration $\vec{g}$ independent of their mass because the gravitational force is proportional to the (gravitational) mass, whereas the acceleration varies inversely with the (inertial) mass. Consider a compartment in space undergoing a uniform acceleration $\vec{a}$, as shown in Figure 39-14*a*. No mechanics experiment can be performed *inside* the compartment that will distinguish whether the compartment is actually accelerating in space or is at rest (or is moving with uniform velocity) in the presence of a uniform gravitational field $\vec{g} = -\vec{a}$, as shown in Figure 39-14*b*. If objects are dropped in the compartment, they will fall to the floor with an acceleration $\vec{g} = -\vec{a}$. If people stand on a spring scale, it will read their weight of magnitude *ma*.

Einstein assumed that the principle of equivalence applies to all physics and not just to mechanics. In effect, he assumed that there is no experiment of any kind that can distinguish uniformly accelerated motion from the presence of a gravitational field.

One consequence of the principle of equivalence—the deflection of a light beam in a gravitational field—was one of the first to be tested experimentally. In a region

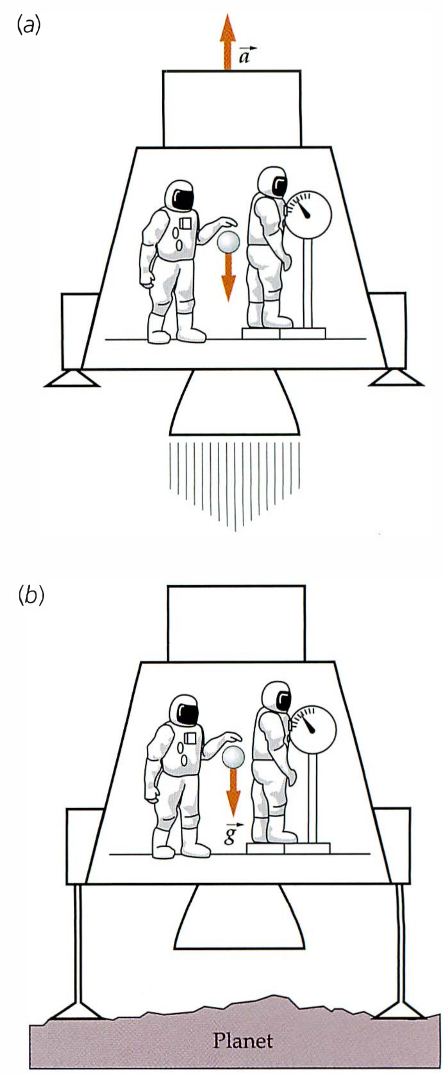


FIGURE 39-14 The results of experiments in a uniformly accelerated reference frame (*a*) cannot be distinguished from those in a uniform gravitational field (*b*) if the acceleration $\vec{a}$ and the gravitational field $\vec{g}$ have the same magnitude.

with no gravitational field, a light beam will travel in a straight line at speed c. The principle of equivalence tells us that a region with no gravitational field exists only in a compartment that is in free fall. Figure 39-15 shows a beam of light entering a compartment that is accelerating relative to a nearby reference frame in free fall. Successive positions of the compartment at equal time intervals are shown in Figure 39-15*a*. Because the compartment is accelerating, the distance it moves in each time interval increases with time. The path of the beam of light as observed from inside the compartment is therefore a parabola, as shown in Figure 39-15*b*. But according to the principle of equivalence, there is no way to distinguish between an accelerating compartment and one moving with uniform velocity in a uniform gravitational field. We conclude, therefore, that a beam of light will accelerate in a gravitational field, just like objects that have mass. For example, near the surface of the earth, light will fall with an acceleration of 9.81 m/s^2. This is difficult to observe because of the enormous speed of light. In a distance of 3000 km, which takes light about 0.01 s to traverse, a beam of light should fall approximately 0.5 mm. Einstein pointed out that the deflection of a light beam in a gravitational field might be observed when light from a distant star passes close to the sun, as illustrated in Figure 39-16. Because of the brightness of the sun, this cannot ordinarily be seen. Such a deflection was first observed in 1919 during an eclipse of the sun. This well-publicized observation brought instant worldwide fame to Einstein.

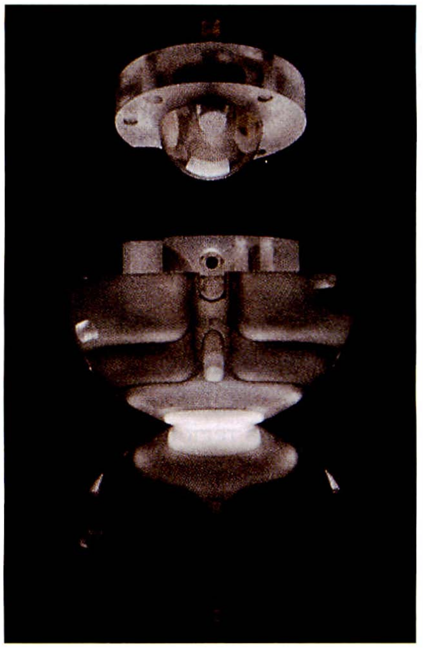
A second prediction from Einstein's theory of general relativity, which we will not discuss in detail, is the excess precession of the perihelion of the orbit of Mercury of about 0.01° per century. This effect had been known and unexplained for some time, so, in a sense, explaining it constituted an immediate success of the theory.

A third prediction of general relativity concerns the change in time intervals and frequencies of light in a gravitational field. In Chapter 11, we found that the gravitational potential energy between two masses M and m a distance r apart is

$$U = -\frac{GMm}{r}$$

where G is the universal gravitational constant, and the point of zero potential energy has been chosen to be when the separation of the masses is infinite. The potential energy per unit mass near a mass M is called the *gravitational potential* ϕ:

$$\phi = -\frac{GM}{r} \qquad 39\text{-}30$$



The quartz sphere in the top part of the container is probably the world's most perfectly round object. It is designed to spin as a gyroscope in a satellite orbiting the earth. General relativity predicts that the rotation of the earth will cause the axis of rotation of the gyroscope to precess in a circle at a rate of approximately 1 revolution in 100,000 years.

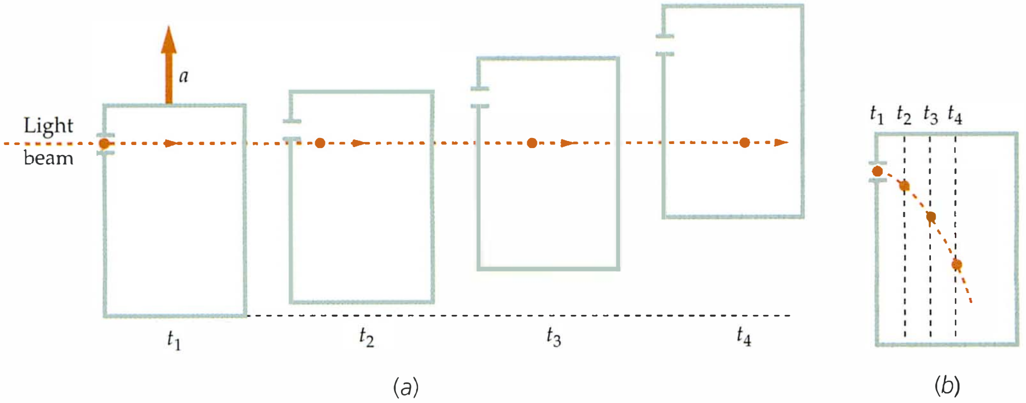


FIGURE 39-15 (*a*) A light beam moving in a straight line through a compartment that is undergoing uniform acceleration relative to a nearby reference frame in free fall. The position of the beam is shown at equally spaced times t_1, t_2, t_3, and t_4. (*b*) In the reference frame of the compartment, the light travels in a parabolic path as a ball would if it were projected horizontally. The vertical displacements are greatly exaggerated in Figure 39-15*a* and Figure 39-15*b* for emphasis.

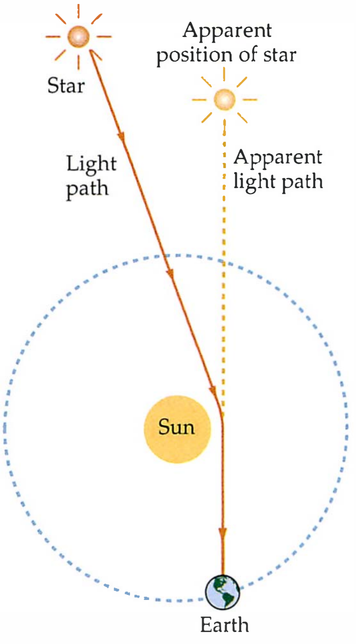
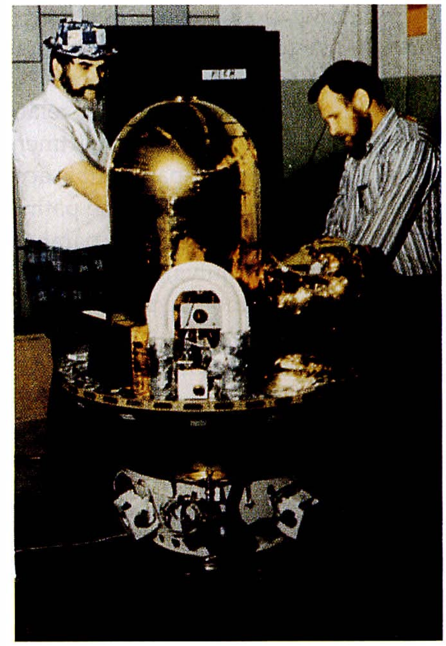


FIGURE 39-16 The deflection (greatly exaggerated) of a beam of light due to the gravitational attraction of the sun.

According to the general theory of relativity, clocks run more slowly in regions of lower gravitational potential. (Since the gravitational potential is negative, as can be seen from Equation 39-30, the nearer the mass the more negative, and therefore the lower the gravitational potential.) If Δt_1 is a time interval between two events measured by a clock where the gravitational potential is ϕ_1 and Δt_2 is the interval between the same events as measured by a clock where the gravitational potential is ϕ_2, general relativity predicts that the fractional difference between these times will be approximately†

$$\frac{\Delta t_2 - \Delta t_1}{\Delta t} = \frac{1}{c^2}(\phi_2 - \phi_1) \qquad \text{39-31}$$

A clock in a region of low gravitational potential will therefore run slower than a clock in a region of high potential. Since a vibrating atom can be considered to be a clock, the frequency of vibration of an atom in a region of low potential, such as near the sun, will be lower than the frequency of vibration of the same atom on the earth. This shift toward a lower frequency, and therefore a longer wavelength, is called the **gravitational redshift.**



This extremely accurate hydrogen maser clock was launched in a satellite in 1976, and its time was compared to that of an identical clock on the earth. In accordance with the prediction of general relativity, the clock on the earth, where the gravitational potential was lower, *lost* approximately 4.3×10^{-10} s each second compared with the clock orbiting the earth at an altitude of approximately 10,000 km.

As our final example of the predictions of general relativity, we mention **black holes,** which were first predicted by J. Robert Oppenheimer and Hartland Snyder in 1939. According to the general theory of relativity, if the density of an object such as a star is great enough, its gravitational attraction will be so great that once inside a critical radius, nothing can escape, not even light or other electromagnetic radiation. (The effect of a black hole on objects outside the critical radius is the same as that of any other mass.) A remarkable property of such an object is that nothing that happens inside it can be communicated to the outside. As sometimes occurs in physics, a simple but incorrect calculation gives the correct results for the relation between the mass and the critical radius of a black hole. In Newtonian mechanics, the speed needed for a particle to escape from the surface of a planet or a star of mass M and radius R is given by Equation 11-21:

$$v_e = \sqrt{\frac{2GM}{R}}$$

If we set the escape speed equal to the speed of light and solve for the radius, we obtain the critical radius R_S, called the **Schwarzschild radius:**

$$R_S = \frac{2GM}{c^2} \qquad \text{39-32}$$

For an object with a mass equal to five times that of our sun (theoretically the minimum mass for a black hole) to be a black hole, its radius would have to be approximately 15 km. Since no radiation is emitted from a black hole and its radius is expected to be small, the detection of a black hole is not easy. The best chance of detection occurs if a black hole is a close companion to a normal star in a binary star system. Then both stars revolve around their center of mass and the gravitational field of the black hole will pull gas from the normal star into the black hole. However, to conserve angular momentum, the gas does not go straight into the black hole. Instead, the gas orbits around the black hole in a disk, called an accretion disk, while slowly being pulled closer to the black hole. The gas in this disk emits X rays because the temperature of the gas being pulled inward reaches several millions of kelvins. The mass of a black-hole candidate can often be estimated. An estimated mass of at least five solar masses, along with the emission of X rays, establishes a strong inference that the candidate is, in fact, a black hole. In addition to the black holes just described, there are supermassive black holes that exist at the centers of galaxies. At the center of the Milky Way is a supermassive black hole with a mass of about two million solar masses.

† Since this shift is usually very small, it does not matter by which interval we divide on the left side of the equation.

SUMMARY

Topic	Relevant Equations and Remarks	
1. **Einstein's Postulates**	The special theory of relativity is based on two postulates of Albert Einstein. All of the results of special relativity can be derived from these postulates. Postulate 1: Absolute uniform motion cannot be detected. Postulate 2: The speed of light is independent of the motion of the source. An important implication of these postulates is Postulate 2 (alternate): Every observer measures the same value c for the speed of light.	
2. **The Lorentz Transformation**	$x = \gamma(x' + vt'), \quad y = y', \quad z = z'$	39-9
	$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$	39-10
	$\gamma = \frac{1}{\sqrt{1 - (v^2/c^2)}}$	39-7
Inverse transformation	$x' = \gamma(x - vt), \quad y' = y, \quad z' = z$	39-11
	$t' = \gamma\left(t - \frac{vx}{c^2}\right)$	39-12
3. **Time Dilation**	The time interval measured between two events that occur at the same point in space in some reference frame is called the proper time t_p. In another reference frame in which the events occur at different places, the time interval between the events is longer by the factor γ.	
	$\Delta t = \gamma \Delta t_p$	39-13
4. **Length Contraction**	The length of an object measured in the reference frame in which the object is at rest is called its proper length L_p. When measured in another reference frame, the length of the object is	
	$L = \frac{L_p}{\gamma}$	39-14
5. **The Relativistic Doppler Effect**	$f' = \frac{\sqrt{1 - (v^2/c^2)}}{1 - (v/c)} f_0$, approaching	39-16*a*
	$f' = \frac{\sqrt{1 - (v^2/c^2)}}{1 + (v/c)} f_0$, receding	39-16*b*
6. **Clock Synchronization and Simultaneity**	Two events that are simultaneous in one reference frame typically are not simultaneous in another frame that is moving relative to the first. If two clocks are synchronized in the frame in which they are at rest, they will be out of synchronization in another frame. In the frame in which they are moving, the chasing clock leads by an amount	
	$\Delta t_S = L_p \frac{v}{c^2}$	39-17
	where L_p is the proper distance between the clocks.	
7. **The Velocity Transformation**	$u_x = \frac{u'_x + v}{1 + (vu'_x/c^2)}$	39-18*a*

	$u_y = \dfrac{u'_y}{\gamma[1 + (vu'_x/c^2)]}$	39-18*b*
	$u_z = \dfrac{u'_z}{\gamma[1 + (vu'_x/c^2)]}$	39-18*c*
Inverse velocity transformation	$u'_x = \dfrac{u_x - v}{1 - (vu_x/c^2)}$	39-19*a*
	$u'_y = \dfrac{u_y}{\gamma[1 - (vu_x/c^2)]}$	39-19*b*
	$u'_z = \dfrac{u_z}{\gamma[1 - (vu_x/c^2)]}$	39-19*c*
8. Relativistic Momentum	$\vec{p} = \dfrac{m\vec{u}}{\sqrt{1 - (u^2/c^2)}}$ where m is the mass of the particle.	39-20
9. Relativistic Energy		
Kinetic energy	$K = \dfrac{mc^2}{\sqrt{1 - (u^2/c^2)}} - mc^2 = \dfrac{mc^2}{\sqrt{1 - (u^2/c^2)}} - E_0$	39-22
Rest energy	$E_0 = mc^2$	39-23
Total energy	$E = K + E_0 = \dfrac{mc^2}{\sqrt{1 - (u^2/c^2)}}$	39-24
10. Useful Formulas for Speed, Energy, and Momentum	$\dfrac{u}{c} = \dfrac{pc}{E}$	39-25
	$E^2 = p^2c^2 + (mc^2)^2$	39-27
	$E \approx pc, \quad \text{for } E >> mc^2$	39-28

PROBLEMS

- • Single-concept, single-step, relatively easy
- •• Intermediate-level, may require synthesis of concepts
- ••• Challenging
- SSM Solution is in the *Student Solutions Manual*
- iSOLVE Problems available on iSOLVE online homework service
- iSOLVE ✓ These "Checkpoint" online homework service problems ask students additional questions about their confidence level, and how they arrived at their answer.

In a few problems, you are given more data than you actually need; in a few other problems, you are required to supply data from your general knowledge, outside sources, or informed estimates.

Conceptual Problems

1 • SSM The approximate total energy of a particle of mass m moving at speed $u << c$ is (*a*) $mc^2 + \frac{1}{2}mu^2$. (*b*) $\frac{1}{2}mu^2$. (*c*) cmu. (*d*) mc^2. (*e*) $\frac{1}{2}cmu$.

2 • SSM A set of twins work in an office building. One twin works on the top floor and the other twin works in the basement. Considering general relativity, which twin will age more quickly? (*a*) They will age at the same rate. (*b*) The twin who works on the top floor will age more quickly. (*c*) The twin who works in the basement will age more quickly. (*d*) It depends on the speed of the office building. (*e*) None of these is correct.

3 • True or false:

(*a*) The speed of light is the same in all reference frames.

(*b*) Proper time is the shortest time interval between two events.
(*c*) Absolute motion can be determined by means of length contraction.
(*d*) The light-year is a unit of distance.
(*e*) Simultaneous events must occur at the same place.
(*f*) If two events are not simultaneous in one frame, they cannot be simultaneous in any other frame.
(*g*) If two particles are tightly bound together by strong attractive forces, the mass of the system is less than the sum of the masses of the individual particles when separated.

4 • An observer sees a system consisting of a mass oscillating on the end of a spring moving past at a speed u and notes that the period of the system is T. Another observer, who is moving with the mass–spring system, also measures its period. The second observer will find a period that is (*a*) equal to T. (*b*) less than T. (*c*) greater than T. (*d*) either (*a*) or (*b*) depending on whether the system was approaching or receding from the first observer. (*e*) There is not sufficient information to answer the question.

5 • The Lorentz transformation for y and z is the same as the classical result: $y = y'$ and $z = z'$. Yet the relativistic velocity transformation does not give the classical result $u_y = u'_y$ and $u_z = u'_z$. Explain.

Estimation and Approximation

6 •• The sun radiates energy at the rate of approximately 4×10^{26} W. Assume that this energy is produced by a reaction whose net result is the fusion of 4 H nuclei to form 1 He nucleus, with the release of 25 MeV for each He nucleus formed. Calculate the sun's loss of mass per day.

7 •• SSM The most distant galaxies that can be seen by the Hubble telescope are moving away from us with a redshift parameter of about $z = 5$. (See Problem 30 for a definition of z.) (*a*) What is the velocity of these galaxies relative to us (expressed as a fraction of the speed of light)? (*b*) *Hubble's law* states that the recession velocity is given by the expression $v = Hx$, where v is the velocity of recession, x is the distance, and H is the Hubble constant, $H = 75$ km/s/Mpc. (1 pc = 3.26 c·y.) Estimate the distance of such a galaxy using the information given.

Time Dilation and Length Contraction

8 • The proper mean lifetime of a muon is 2 μs. Muons in a beam are traveling through a laboratory at $0.95c$. (*a*) What is their mean lifetime as measured in the laboratory? (*b*) How far do they travel, on average, before they decay?

9 •• In the Stanford linear collider, small bundles of electrons and positrons are fired at each other. In the laboratory's frame of reference, each bundle is approximately 1 cm long and 10 μm in diameter. In the collision region, each particle has an energy of 50 GeV, and the electrons and the positrons are moving in opposite directions. (*a*) How long and how wide is each bundle in its own reference frame? (*b*) What must be the minimum proper length of the accelerator for a bundle to have both its ends simultaneously in the accelerator in its own reference frame? (The actual length of the accelerator is less than 1000 m.) (*c*) What is the length of a positron bundle in the reference frame of the electron bundle?

10 •• SSM Unobtainium (Un) is an unstable particle that decays into normalium (Nr) and standardium (St) particles. (*a*) An accelerator produces a beam of Un that travels to a detector located 100 m away from the accelerator. The particles travel with a velocity of $v = 0.866c$. How long do the particles take (in the laboratory frame) to get to the detector? (*b*) By the time the particles get to the detector, half of the particles have decayed. What is the half-life of Un? (*Note:* Half-life as it would be measured in a frame moving with the particles.) (*c*) A new detector is going to be used, which is located 1000 m away from the accelerator. How fast should the particles be moving if half of the particles are to make it to the new detector?

11 •• Star A and Star B are at rest relative to the earth. Star A is 27 c·y from earth, and Star B is located beyond (behind) Star A as viewed from earth. (*a*) A spaceship is making a trip from earth to Star A at a speed such that the trip from earth to Star A takes 12 y according to clocks on the spaceship. At what speed, relative to earth, must the ship travel? (Assume that the times for acceleration are very short compared to the overall trip time.) (*b*) Upon reaching Star A, the ship speeds up and departs for Star B at a speed such that the gamma factor, γ, is twice that of Part (*a*). The trip from Star A to Star B takes 5 y (ship's time). How far, in c·y, is Star B from Star A in the rest frame of the earth and the two stars? (*c*) Upon reaching Star B, the ship departs for earth at the same speed as in Part (*b*). It takes it 10 y (ship's time) to return to earth. If you were born on earth the day the ship left earth (and you remain on earth), how old are you on the day the ship returns to earth?

12 • A spaceship travels to a star 35 c·y away at a speed of 2.7×10^8 m/s. How long does the spaceship take to get to the star (*a*) as measured on the earth and (*b*) as measured by a passenger on the spaceship?

13 • Use the binomial expansion equation

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \ldots \approx 1 + nx, \quad \text{for } x << 1$$

to derive the following results for the case when v is much less than c.

(*a*) $\gamma \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$

(*b*) $\frac{1}{\gamma} \approx 1 - \frac{1}{2}\frac{v^2}{c^2}$

(*c*) $\gamma - 1 \approx 1 - \frac{1}{\gamma} \approx \frac{1}{2}\frac{v^2}{c^2}$

14 •• A clock on Spaceship A measures the time interval between two events, both of which occur at the location of the clock. You are on Spaceship B. According to your careful measurements, the time interval between the two events is 1 percent longer than that measured by the two clocks on Spaceship A. How fast is Ship A moving relative to Ship B. (Use one or more of the results of Problem 13.)

15 •• If a plane flies at a speed of 2000 km/h, how long must the plane fly before its clock loses 1 s because of time dilation? (Use one or more of the results of Problem 13.)

The Lorentz Transformation, Clock Synchronization, and Simultaneity

16 •• Show that when $v << c$ the transformation equations for x, t, and u reduce to the Galilean equations.

17 •• SSM ISOLVE A spaceship of proper length L_p = 400 m moves past a transmitting station at a speed of 0.76c. At the instant that the nose of the spaceship passes the transmitter, clocks at the transmitter and in the nose of the spaceship are synchronized to $t = t' = 0$. The instant that the tail of the spaceship passes the transmitter a signal is sent and subsequently detected by the receiver in the nose of the spaceship. (*a*) When, according to the clock in the spaceship, is the signal sent? (*b*) When, according to the clock at the transmitter, is the signal received by the spaceship? (*c*) When, according to the clock in the spaceship, is the signal received? (*d*) Where, according to an observer at the transmitter, is the nose of the spaceship when the signal is received?

18 •• In frame S, event B occurs 2 μs after event A, which occurs at x = 1.5 km from event A. How fast must an observer be moving along the $+x$ axis so that events A and B occur simultaneously? Is it possible for event B to precede event A for some observer?

19 •• Observers in reference frame S see an explosion located at x_1 = 480 m. A second explosion occurs 5 μs later at x_2 = 1200 m. In reference frame S', which is moving along the $+x$ axis at speed v, the explosions occur at the same point in space. What is the separation in time between the two explosions as measured in S'?

20 ••• Two events in S are separated by a distance $D = x_2 - x_1$ and a time $T = t_2 - t_1$. (*a*) Use the Lorentz transformation to show that in frame S', which is moving with speed v relative to S, the time separation is $t_2' - t_1' = \gamma(T - vD/c^2)$. (*b*) Show that the events can be simultaneous in frame S' only if D is greater than cT. (*c*) If one of the events is the *cause* of the other, the separation D must be less than cT, since D/c is the smallest time that a signal can take to travel from x_1 to x_2 in frame S. Show that if D is less than cT, t_2' is greater than t_1' in all reference frames. This shows that if the cause precedes the effect in one frame, it must precede it in all reference frames. (*d*) Suppose that a signal could be sent with speed $c' > c$ so that in frame S the cause precedes the effect by the time $T = D/c'$. Show that there is then a reference frame moving with speed v less than c in which the effect precedes the cause.

21 ••• A rocket with a proper length of 700 m is moving to the right at a speed of 0.9c. It has two clocks, one in the nose and one in the tail, that have been synchronized in the frame of the rocket. A clock on the ground and the nose clock on the rocket both read $t = 0$ as they pass. (*a*) At $t = 0$, what does the tail clock on the rocket read as seen by an observer on the ground? When the tail clock on the rocket passes the ground clock, (*b*) what does the tail clock read as seen by an observer on the ground, (*c*) what does the nose clock read as seen by an observer on the ground, and (*d*) what does the nose clock read as seen by an observer on the rocket? (*e*) At t = 1 h, as measured on the rocket, a light signal is sent from the nose of the rocket to an observer standing by the ground clock. What does the ground clock read when the observer receives this signal? (*f*) When the observer on the ground receives the signal, he sends a return signal to the nose of the rocket. When is this signal received at the nose of the rocket as seen on the rocket?

22 ••• SSM An observer in frame S standing at the origin observes two flashes of colored light separated spatially by Δx = 2400 m. A blue flash occurs first, followed by a red flash 5 μs later. An observer in S' moving along the x axis at speed v relative to S also observes the flashes 5 μs apart and with a separation of 2400 m, but the red flash is observed first. Find the magnitude and direction of v.

The Velocity Transformation

23 •• Show that if u_x' and v in Equation 39-18*a* are both positive and less than c, then u_x is positive and less than c. [*Hint: Let* $u_x' = (1 - \varepsilon_1)c$ *and* $v = (1 - \varepsilon_2)c$, *where* ε_1 *and* ε_2 *are positive numbers that are less than 1.*]

24 •• SSM A spaceship, at rest in a certain reference frame S, is given a speed increment of 0.50c (call this boost 1). Relative to its new rest frame, the spaceship is given a further 0.50c increment 10 seconds later (as measured in its new rest frame; call this boost 2). This process is continued indefinitely, at 10-s intervals, as measured in the rest frame of the ship. (Assume that the boost itself takes a very short time compared to 10 s.) (*a*) Using a spreadsheet program, calculate and graph the velocity of the spaceship in reference frame S as a function of the boost number for boost 1 to boost 10. (*b*) Graph the gamma factor the same way. (*c*) How many boosts does it take until the velocity of the ship in S is greater than 0.999c? (*d*) How far has the spaceship moved after 5 boosts, as measured in reference frame S? What is the average speed of the spaceship (between boost 1 and boost 5) as measured in S?

The Relativistic Doppler Effect

25 • Sodium light of wavelength 589 nm is emitted by a source that is moving toward the earth with speed v. The wavelength measured in the frame of the earth is 547 nm. Find v.

26 • A distant galaxy is moving away from us at a speed of 1.85×10^7 m/s. Calculate the fractional redshift $(\lambda' - \lambda_0)/\lambda_0$ in the light from this galaxy.

27 •• Derive Equation 39-16*a* for the frequency received by an observer moving with speed v toward a stationary source of electromagnetic waves.

28 • Show that if v is much less than c, the Doppler shift is given approximately by

$$\Delta f/f \approx \pm v/c$$

29 •• SSM A clock is placed in a satellite that orbits the earth with a period of 90 min. By what time interval will this clock differ from an identical clock on the earth after 1 y? (Assume that special relativity applies and neglect general relativity.)

30 •• For light that is Doppler-shifted with respect to an observer, define the redshift parameter

$$z = \frac{f - f'}{f'}$$

where f is the frequency of the light measured in the rest frame of the emitter, and f' is the frequency measured in the rest frame of the observer. If the emitter is moving directly away from the observer, show that the relative velocity between the emitter and the observer is

$$v = c\left(\frac{u^2 - 1}{u^2 + 1}\right)$$

where $u = z + 1$.

31 • A light beam moves along the y' axis with speed c in frame S', which is moving to the right with speed v relative to frame S. (*a*) Find the x and y components of the velocity of the light beam in frame S. (*b*) Show that the magnitude of the velocity of the light beam in S is c.

32 • A spaceship is moving east at speed $0.90c$ relative to the earth. A second spaceship is moving west at speed $0.90c$ relative to the earth. What is the speed of one spaceship relative to the other spaceship?

33 •• SSM A particle moves with speed $0.8c$ along the x'' axis of frame S'', which moves with speed $0.8c$ along the x' axis relative to frame S'. Frame S' moves with speed $0.8c$ along the x axis relative to frame S. (*a*) Find the speed of the particle relative to frame S'. (*b*) Find the speed of the particle relative to frame S.

Relativistic Momentum and Relativistic Energy

34 • SSM A proton (rest energy 938 MeV) has a total energy of 2200 MeV. (*a*) What is its speed? (*b*) What is its momentum?

35 • If the kinetic energy of a particle equals twice its rest energy, what percentage error is made by using $p = mu$ for its momentum?

36 •• ISOLVE✓ A particle with momentum of 6 MeV/c has total energy of 8 MeV. (*a*) Determine the mass of the particle. (*b*) What is the energy of the particle in a reference frame in which its momentum is 4 MeV/c? (*c*) What are the relative velocities of the two reference frames?

37 •• Show that

$$d\left(\frac{mu}{\sqrt{1 - (u^2/c^2)}}\right) = m\left(1 - \frac{u^2}{c^2}\right)^{-3/2} du$$

38 •• ISOLVE✓ The K^0 particle has a mass of 497.7 MeV/c^2. It decays into a π^- and π^+, each with mass 139.6 MeV/c^2. Following the decay of a K^0, one of the pions is at rest in the laboratory. Determine the kinetic energy of the other pion and of the K^0 prior to the decay.

39 •• SSM Two protons approach each other head-on at $0.5c$ relative to reference frame S'. (*a*) Calculate the total kinetic energy of the two protons as seen in frame S'. (*b*) Calculate the total kinetic energy of the protons as seen in reference frame S, which is moving with speed $0.5c$ relative to S' so that one of the protons is at rest.

40 •• An antiproton has the same rest energy as a proton. It is created in the reaction $p + p \to p + p + p + \bar{p}$. In an experiment, protons at rest in the laboratory are bombarded with protons of kinetic energy K_L, which must be great enough so that kinetic energy equal to $2mc^2$ can be converted into the rest energy of the two particles. In the frame of the laboratory, the total kinetic energy cannot be converted into rest energy because of conservation of momentum. However, in the zero-momentum reference frame in which the two initial protons are moving toward each other with equal speed u, the total kinetic energy can be converted into rest energy. (*a*) Find the speed of each proton u so that the total kinetic energy in the zero-momentum frame is $2mc^2$. (*b*) Transform to the laboratory's frame in which one proton is at rest, and find the speed u' of the other proton. (*c*) Show that the kinetic energy of the moving proton in the laboratory's frame is $K_L = 6mc^2$.

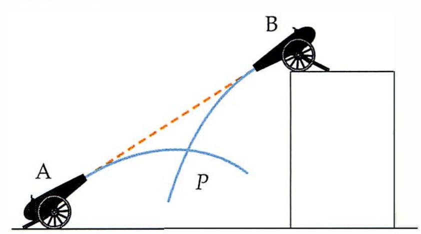
41 ••• ISOLVE A particle of mass 1 MeV/c^2 and kinetic energy 2 MeV collides with a stationary particle of mass 2 MeV/c^2. After the collision, the particles stick together. Find (*a*) the speed of the first particle before the collision, (*b*) the total energy of the first particle before the collision, (*c*) the initial total momentum of the system, (*d*) the total kinetic energy after the collision, and (*e*) the mass of the system after the collision.

General Relativity

42 •• SSM Light traveling in the direction of increasing gravitational potential undergoes a frequency redshift. Calculate the shift in wavelength if a beam of light of wavelength $\lambda = 632.8$ nm is sent up a vertical shaft of height $L = 100$ m.

43 •• Let us revisit a problem from Chapter 3: Two cannons are pointed directly toward each other, as shown in Figure 39-17. When fired, the cannonballs will follow the trajectories shown. Point P is the point where the trajectories cross each other. Ignore the effects of air resistance. Using the principle of equivalence, show that if the cannons are fired simultaneously, the cannonballs will hit each other at point P.

FIGURE 39-17 Problem 43



44 ••• A horizontal turntable rotates with angular speed ω. There is a clock at the center of the turntable and one at a distance r from the center. In an inertial reference frame, the clock at distance r is moving with speed $u = r\omega$. (*a*) Show that from time dilation according to special relativity, time intervals Δt_0 for the clock at rest and Δt_r for the moving clock are related by

$$\frac{\Delta t_r - \Delta t_0}{\Delta t_0} = -\frac{r^2\omega^2}{2c^2}, \qquad \text{if } r\omega << c$$

(*b*) In a reference frame rotating with the table, both clocks are at rest. Show that the clock at distance r experiences a pseudoforce $F_r = mr\omega^2$ in this accelerated frame and that this is equivalent to a difference in gravitational potential between r and the origin of $\phi_r - \phi_0 = -\frac{1}{2}r^2\omega^2$. Use this potential difference given in Part (b) to show that in this frame the difference in time intervals is the same as in the inertial frame.

General Problems

45 • How fast must a muon travel so that its mean lifetime is 46 μs if its mean lifetime at rest is 2 μs?

46 • ISOLVE✓ A distant galaxy is moving away from the earth with a speed that results in each wavelength received on the earth being shifted so that $\lambda' = 2\lambda_0$. Find the speed of the galaxy relative to the earth.

47 •• SSM Frames S and S' are moving relative to each other along the x and x' axes. Observers in the two frames set their clocks to $t = 0$ when the origins coincide. In frame S, event 1 occurs at $x_1 = 1.0\ c{\cdot}\text{y}$ and $t_1 = 1$ y and event 2 occurs at $x_2 = 2.0\ c{\cdot}\text{y}$ and $t_2 = 0.5$ y. These events occur simultaneously in frame S'. (a) Find the magnitude and direction of the velocity of S' relative to S. (b) At what time do both these events occur as measured in S'?

48 •• An interstellar spaceship travels from the earth to a distant star system 12 light-years away (as measured in the earth's frame). The trip takes 15 y as measured on the spaceship. (a) What is the speed of the spaceship relative to the earth? (b) When the ship arrives, it sends a signal to the earth. How long after the ship leaves the earth will it be before the earth receives the signal?

49 •• The neutral pion π^0 has a mass of 135 MeV$/c^2$. This particle can be created in a proton–proton collision:

$$\mathrm{p} + \mathrm{p} \rightarrow \mathrm{p} + \mathrm{p} + \pi^0$$

Determine the threshold kinetic energy for the creation of a π^0 in a collision of a moving proton and a stationary proton. (See Problem 40.)

50 •• A rocket with a proper length of 1000 m moves in the $+x$ direction at $0.6c$ with respect to an observer on the ground. An astronaut stands at the rear of the rocket and fires a bullet toward the front of the rocket at $0.8c$ relative to the rocket. How long does it take the bullet to reach the front of the rocket (a) as measured in the frame of the rocket, (b) as measured in the frame of the ground, and (c) as measured in the frame of the bullet?

51 ••• SSM In a simple thought experiment, Einstein showed that there is mass associated with electromagnetic radiation. Consider a box of length L and mass M resting on a frictionless surface. At the left wall of the box is a light source that emits radiation of energy E, which is absorbed at the right wall of the box. According to classical electromagnetic theory, this radiation carries momentum of magnitude $p = E/c$ (Equation 32-13). (a) Find the recoil velocity of the box so that momentum is conserved when the light is emitted. (Since p is small and M is large, you may use classical mechanics.) (b) When the light is absorbed at the right wall of the box the box stops, so the total momentum remains zero. If we neglect the very small velocity of the box, the time it takes for the radiation to travel across the box is $\Delta t = L/c$. Find the distance moved by the box in this time. (c) Show that if the center of mass of the system is to remain at the same place, the radiation must carry mass $m = E/c^2$.

52 ••• Reference frame S' is moving along the x' axis at $0.6c$ relative to frame S. A particle that is originally at $x' = 10$ m at $t_1' = 0$ is suddenly accelerated and then moves at a constant speed of $c/3$ in the $-x'$ direction until time $t_2' = 60$ m$/c$, when it is suddenly brought to rest. As observed in frame S, find (a) the speed of the particle, (b) the distance and the direction that the particle traveled from t_1' to t_2', and (c) the time the particle traveled.

53 ••• In reference frame S, the acceleration of a particle is $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$. Derive expressions for the acceleration components a_x', a_y', and a_z' of the particle in reference frame S' that is moving relative to S in the x direction with velocity v.

54 ••• Using the relativistic conservation of momentum and energy and the relation between energy and momentum for a photon $E = pc$, prove that a free electron (i.e., one not bound to an atomic nucleus) cannot absorb or emit a photon.

55 ••• SSM When a projectile particle with kinetic energy greater than the threshold kinetic energy K_{th} strikes a stationary target particle, one or more particles may be created in the inelastic collision. Show that the threshold kinetic energy of the projectile is given by

$$K_{\text{th}} = \frac{(\Sigma m_{\text{in}} + \Sigma m_{\text{fin}})(\Sigma m_{\text{fin}} - \Sigma m_{\text{in}})c^2}{2m_{\text{target}}}$$

Here Σm_{in} is the sum of the masses of the projectile and target particles, Σm_{fin} is the sum of the masses of the final particles, and m_{target} is the mass of the target particle. Use this expression to determine the threshold kinetic energy of protons incident on a stationary proton target for the production of a proton–antiproton pair; compare your result with the result of Problem 40.

56 ••• A particle of mass M decays into two identical particles of mass m, where $m = 0.3M$. Prior to the decay, the particle of mass M has an energy of $4Mc^2$ in the laboratory. The velocities of the decay products are along the direction of motion of M. Find the velocities of the decay products in the laboratory.

57 ••• A stick of proper length L_{p} makes an angle θ with the x axis in frame S. Show that the angle θ' made with the x' axis in frame S', which is moving along the $+x$ axis with speed v, is given by $\tan\theta' = \gamma\tan\theta$ and that the length of the stick in S' is

$$L' = L_{\text{p}}\left(\frac{1}{\gamma^2}\cos^2\theta + \sin^2\theta\right)^{1/2}$$

58 ••• Show that if a particle moves at an angle θ with the x axis with speed u in frame S, it moves at an angle θ' with the x' axis in S' given by

$$\tan\theta' = \frac{\sin\theta}{\gamma[\cos\theta - (v/u)]}$$

59 ••• SSM For the special case of a particle moving with speed u along the y axis in frame S, show that its momentum and energy in frame S', a frame that is moving along the x axis with velocity v, are related to its momentum and energy in S by the transformation equations

$$p'_x = \gamma\left(p_x - \frac{vE}{c^2}\right), \qquad p'_y = p_y, \qquad p'_z = p_z$$

$$\frac{E'}{c} = \gamma\left(\frac{E}{c} - \frac{vp_x}{c}\right)$$

Compare these equations with the Lorentz transformation for x', y', z', and t'. These equations show that the quantities p_x, p_y, p_z, and E/c transform in the same way as do x, y, z, and ct.

60 ••• The equation for the spherical wavefront of a light pulse that begins at the origin at time $t = 0$ is $x^2 + y^2 + z^2 - (ct)^2 = 0$. Using the Lorentz transformation, show that such a light pulse also has a spherical wavefront in frame S' by showing that $x'^2 + y'^2 + z'^2 - (ct')^2 = 0$ in S'.

61 ••• In Problem 60, you showed that the quantity $x^2 + y^2 + z^2 - (ct)^2$ has the same value (0) in both S and S'. Such a quantity is called an *invariant*. From the results of Problem 59, the quantity $p_x^2 + p_y^2 + p_z^2 - E^2/c^2$ must also be an invariant. Show that this quantity has the value $-m^2c^2$ in both the S and S' reference frames.

62 ••• SSM A long rod that is parallel to the x axis is in free fall with acceleration g parallel to the $-y$ axis. An observer in a rocket moving with speed v parallel to the x axis passes by and watches the rod falling. Using the Lorentz transformations, show that the observer will measure the rod to be bent into a parabolic shape. Is the parabola concave upward or concave downward?