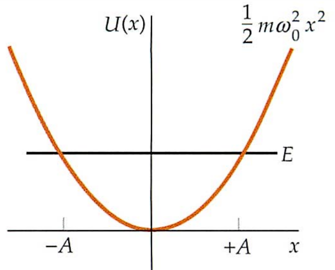
35-3 The Harmonic Oscillator

The potential energy for a particle of mass m attached to a spring of force constant k is

$$U(x) = \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\omega_0^2x^2 \qquad \text{35-19}$$

where $\omega_0 = \sqrt{k/m}$ is the natural frequency of the oscillator. Classically, the object oscillates between $x = +A$ and $x = -A$. The object's total energy is $E = \tfrac{1}{2}m\omega_0^2A^2$, which can have any positive value or zero.



For the harmonic oscillator potential energy function, the Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \tfrac{1}{2}m\omega_0^2x^2\psi(x) = E\psi(x) \qquad \text{35-22}$$

Wave Functions and Energy Levels

Rather than pursue a general solution to the Schrödinger equation for this system, we simply present the solution for the ground state and the first excited state.

The ground-state wave function $\psi_0(x)$ is found to be a Gaussian function centered at the origin:

$$\psi_0(x) = A_0e^{-ax^2} \qquad \text{35-23}$$

where A_0 and a are constants. This function and the wave function for the first excited state are shown in Figure 35-7.

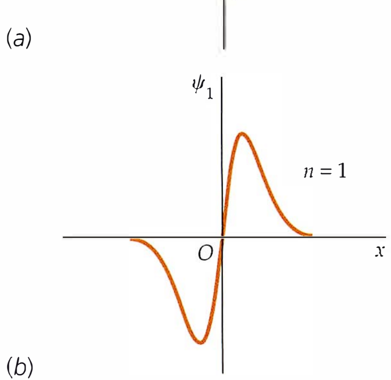


FIGURE 35-7 (*a*) The ground-state wave function for the harmonic oscillator potential. (*b*) The wave function for the first excited state of the harmonic oscillator potential.

We see from this example that the ground-state energy is given by

$$E_0 = \frac{\hbar^2 a}{m} = \frac{1}{2}\hbar\omega_0 \tag{35-24}$$

The first excited state has a node in the center of the potential well, just as with the particle in a box.† The wave function $\psi_1(x)$ is

$$\psi_1(x) = A_1 x e^{-ax^2} \tag{35-25}$$

where $a = m\omega_0/2\hbar$, as in Example 35-1. This function is also shown in Figure 35-7. Substituting $\psi_1(x)$ into the Schrödinger equation, as was done for $\psi_0(x)$ in Example 35-1, yields the energy of the first excited state,

$$E_1 = \tfrac{3}{2}\hbar\omega_0$$

In general, the energy of the nth excited state of the harmonic oscillator is

$$E_n = (n + \tfrac{1}{2})\hbar\omega_0, \quad n = 0, 1, 2, \ldots \tag{35-26}$$

as indicated in Figure 35-8. The fact that the energy levels are evenly spaced by the amount $\hbar\omega_0$ is a peculiarity of the harmonic oscillator potential. As we saw in Chapter 34, the energy levels for a particle in a box, or for the hydrogen atom, are not evenly spaced. The precise spacing of energy levels is closely tied to the particular form of the potential energy function.

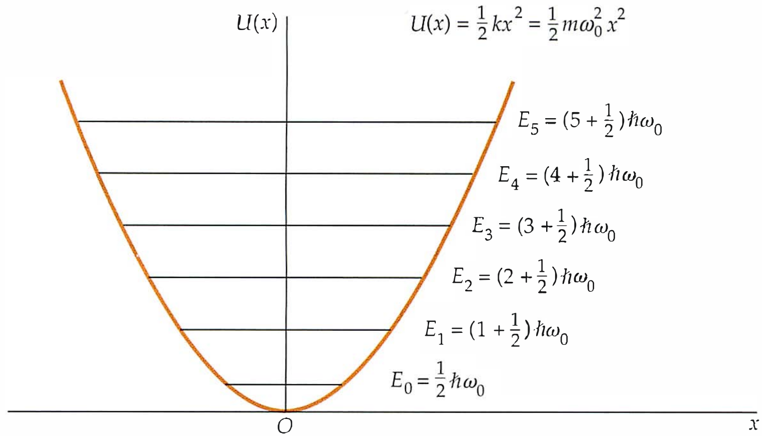


FIGURE 35-8 Energy levels in the harmonic oscillator potential.